существенно или поверхностно сходство, схваченное в метафоре» [20].

Нелинейные эффекты пространственно-временного характера присутствуют в естественных и искусственных системах [21]. Они с заметным постоянством возникают в технике, экономике и организационном управлении в контексте создания высокоэффективных систем, их роль и влияние обсуждаются с растущей интенсивностью в научных трудах отечественных и зарубежных авторов.

ЛИТЕРАТУРА

1. Клир Дж. Системология. Опыт решения системных задач. /Пер. с англ., – М.: Радио и связь, 1990. – 544 с.
2. Холл А. Опыт методологии для системотехники. – М.: Советское радио, 1975. – 607 с.
3. Коэн А., Харкурт Дж. Судьба дискуссии двух Кембриджей о теории капитала // Вопросы экономики, 2009, № 8. – С. 4–27.
4. Клейнер Г. Б. Производственные функции: теория, методы, применение. – М.: Финансы и статистика, 1986. – 239 с.
5. Мэнкью Н. Г. Макроэкономист как ученый и инженер // Вопросы экономики, 2009, № 8. – С. 86–103.
6. Интрилигатор М. Математические методы оптимизации и экономическая теория. – М.: Айрис Пресс, 2002. – 564 с.
7. Ставерен И. Этика эффективности // Вопросы экономики, 2009. № 12. – С. 58–71.
8. Базара М., Шетти К. Нелинейное программирование. Теория и алгоритмы. – М.: Мир, 1982. – 580 с.
9. Обен, Ж. – П. Нелинейный анализ и ее экономические приложения. – М.: Мир, 1988. – 264 с.
10. Сейдж Э. П., Уайт Ч. С., III. Оптимальное управление системами. – М.: Радио и связь, 1982. – 392 с.
11. Дафт Р. Л. Теория организации. – М.: ЮНИТИ-ДАНА, 2006. – 736 с.
12. Демиденко Е. З. Метод динамизации коэффициентов регрессии. //Автоматика и телемехагника, 1987, № 9. – С. 77–83.
13. Экланд И. Элементы математической экономики. – М.: Мир, 1983. – 245 с.
14. Кемпбелл Э., Лачс К. С. Стратегический синергизм. – СПб.: Питер, 2004. – 416 с.
15. Саркисян Р. Е. Предпочтения, полезность и меры чувствительности многокритериальных альтернатив // Экономика и математические методы, 2009, том 45, № 3. – С. 87–105.
16. Аллен Р.Дж.Д. Математическая экономика: Пер. с англ. Под ред. А. Л. Вайнштейна. – М.: Мир, 1963. – 580 с.
17. Друри К. Управленческий и производственный учет. Вводный курс: Учебник. – М.: ЮНИТИ-ДАНА, 2005. – 735 с.
18. Управленческое консультирование. Введение в профессию / Под ред. М. Кубра. – М.: Планум, 2004. – 976 с.
19. Делавинья С. Психология и экономика: Результаты эмпирических исследований. Часть I: Нестандартные предпочтения //Вопросы экономики, 2011, № 4. – С. 50–77.
20. Саймон Г. А. Теория принятия решений в экономической теории и науке о поведении // Теория фирмы. – СПб.: 1995. – С. 54–72..
21. Майнцер К. Сложносистемное мышление: Материя, разум, человечество. Новый синтез. – М.: Книжный дом «Либроком», 2009. – 464 с. ●

DEPENDENCE OF THE USED RESOURCES ON THE NONLINEARITY OF THE PRODUCTION FUNCTION

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ABSTRACT

The inherent nonlinearity of production functions and loss of sensitivity generated by it and decreasing effectiveness are investigated in the framework of sensitivity theory. On this basis, the problem of optimization of resources is explored according to the criteria of benefits and costs, as well as the nonlinear nature of the central causation, which is characteristic of natural and artificial systems. The effects, reflecting these features, are spatio-temporal in nature and appear in engineering, economics, management. Attention to them is only increasing.

ENGLISH SUMMARY

Background

It is generally accepted that the most complete and useful characteristic of any organization can be obtained in the framework of system theory [1, 2]. This is true also with respect to such key aspects of the structure and functioning of the organization as a rational use of resources, productivity, management efficiency, etc. Pic. 1 shows the most common and productive system view of the organization as an open system «input – output».

Traditionally, it is assumed that a static picture of the observed causal relationship between input and output variables can be described quite accurately with the help of the generalized production function $Y = F(q1, \dots ,qn)$ *that characterizes the technological relationship between value (goods and services) created by the organization and the costs incurred by economic factors. And despite a certain controversy, of «two Cambridges « [3], the concept of the production function remains one of the very constructive mathematical schemes and models for economic analysis and forecasting at the macro and micro levels. [4] Of particular value is its «engineering» component, which plays an important role in the theory and techniques of economic analysis and control [5].*

The theory of the firm postulates the existence of a special economic area (convex subsets of costs), at the points of which the production function $Y = F(q_1, \dots , q_n)$ *satisfies the conditions* $\partial F / \partial q_j > 0,\ \partial^2 F / \partial q_j^2 < 0,\ j = 1,\dots,\ n,$ *[6]. The first of* these means that the so-called marginal product $\partial F / \partial q_j, j = 1,..., n$ is positive, but decreases with increasing values of the associated economic factor, which is reflected in the second condition. This fact is known as decreasing returns law (or the law of Gossen).

Causal relationship $Y = F(q_1, ..., q_n)$ has a non-linear nature, which causes loss of sensitivity between inputs and outputs of the system (between impulse and response) and, as a consequence, the loss of efficiency of production factors (or resources). Such nonlinear effects are directly related to the problem of effectiveness ethics in relation to the resources, which in recent years has become the object of study of the growing intensity [7].

In the article the author offers a suitable analytical tool for describing and evaluating already mentioned nonlinear effects associated with the characteristic of $Y = F(q_1, ..., q_n)$ in the framework of sensitivity theory and defines their role and influence on the development of effective relationships for a number of system characteristics of scientific and practical interest.

Methods.

In the first part of the article the author focuses on the measures of sensitivity and decreasing effectiveness. It is assumed that the non-linear relationship $Y = F(q)$, $q = (q_1, ..., q_n)^T$ belongs to the class of concave ascending series on a nonempty convex set $Q \subset E_+^n$ of twice continuously differentiable functions $F: Q \to E_+^1$, which satisfies the conditions $\partial F / \partial q_j > 0,\ \partial^2 F / \partial q_j^2 < 0,\ j = 1,..., n,$ where $q_j, j = 1, ..., n,$ - coordinates of the vector $q \in Q$, E_+^n - the nonnegative orthant of $n-$ dimensional Euclidean space E^n.

From optimization theory it follows that concave ascending series on a nonempty convex set $Q \subset E_+^n$ function $F: Q \to E_+^1$ satisfies the differential inequality [8].

$$F(q) - F(\overline{q}) \le \nabla F(\overline{q})^T (q - \overline{q}),\ q \in Q, \qquad (1.1)$$

where $\overline{q} \in Q -$ arbitrary admissible point, $\nabla F(\overline{q}) -$ gradient of function $F(q)$ at this point. This function $y = F(\overline{q}) + \nabla F(\overline{q})^T (q - \overline{q})$ corresponds to the hyperplane

$$H_F = \{(y, q) / y = F(\overline{q}) + \nabla F(\overline{q})^T (q - \overline{q}),\ q \in Q\},$$

tangent to the surface of the function $F(q)$ at the point $\overline{q}$, and gradient $\nabla F(\overline{q})$ - the slope of the hyperplane. If $F_0(\vec{q})$ is an ordinate of the point of hyperplane H_F intersection with the axis of the function $F(q)$ and noting that $\overline{q} \in Q$ is an arbitrary point, there is an expansion for the function $F(q)$ in the form

$$F(q) = F_0(q) + \nabla F(q)^T q,\ q \in Q. \qquad (1.2)$$

The second summand on the right side is an inner product of two vectors: gradient

$$\nabla F(q) \equiv \frac{\partial F}{\partial q} \equiv \left(\frac{\partial F}{\partial q_1}, ..., \frac{\partial F}{\partial q_n}\right)^T$$

and a vector $q = (q_1, ..., q_n)^T$. The formula for this component $v(q)$, is

$$v(q) \equiv \nabla F(q)^T q = \sum_{j=1}^{n} \frac{\partial F(q)}{\partial q_j} q_j, \qquad (1.3).$$

Perturbation of the argument of the function provides valuable information about its local dynamic properties, including sensitivity. «Caring about the simplicity and effectiveness' [9], the author uses for this purpose ratio limit

$$\frac{dF(tq) / F(tq)}{dt / t} = \frac{t}{F(tq)} \frac{dF(tq)}{dt} \qquad (1.4)$$

where $t \to 1$. If this limit is determined through $\sigma(q)$ and taking into account that $F(tq) = F(tq_1, ..., tq_n)$, and its derivative on the parameter t is

$$\frac{d}{dt} F(tq_1, ..., tq_n) = \sum_{j=1}^{n} \frac{\partial F(tq)}{\partial (tq_j)} \frac{\partial (tq_j)}{\partial t} = \sum_{j=1}^{n} \frac{\partial F(tq)}{\partial (tq_j)} q_j, \qquad (1.5)$$

for $\sigma(q)$ there is a formula

$$\sigma(q) \equiv \lim_{t \to 1} \frac{t}{F(tq)} \frac{dF(tq)}{dt} = \frac{1}{F(q)} \sum_{j=1}^{n} \frac{\partial F(q)}{\partial q_j} q_j. \qquad (1.6)$$

In essence, $\sigma(q)$ represents the local rate of change of a function $F(q)$ from the proportional coordinates change of the vector $q = (q_1, ..., q_n)^T$, and its additive structure differentially reflects the influence of each of the coordinates of the vector.

Thus, $\sigma(q)$ can be considered as an integral measure of (relative) sensitivity of function $F(q)$ at the point $q \in Q$, and its components $\sigma_j(q)$, $j = 1, ..., n$, as particular measures of (relative) sensitivity. In economics, these figures are known as malleability.

Using (1.6), the expression (1.3) can be represented in the form

$$v(q) = \sigma(q) F(q), \qquad (1.7)$$

which, in turn, enables to rewrite formula (1.2) in the form

$$F(q) = F_0(q) + v(q) = (1 - \sigma(q)) F(q) + \sigma(q) F(q),\ q \in Q. \qquad (1.8)$$

Functions $v(f)$ and $\sigma(f)$ have additive structure and their components are interconnected the following way

$$v_j(q) = \frac{\partial F(q)}{\partial q_j} q_j = \sigma_j(q) F(q),\ j = 1, ..., n, \qquad (1.9)$$

In essence, each of the figures $v_j(q)$, $j = 1, ..., n$ is by analogy with the known concept of productivity [11] «partial return» (or "partial productivity") corresponding the variable q_j, $j = 1, ..., n$, at a fixed figure of the other variables. In this respect, $v(q)$ can be interpreted as «cumulative return» (or "cumulative productivity") of input variables $q_1, ..., q_n$. It is important to note that

$$v_i(q) / v_j(q) = \sigma_i(q) / \sigma_j(q) = (q_i / q_j) \mu_{ij},\ i, j = 1, ..., n, \qquad (1.10)$$

where through μ_{ij} so-called marginal substitution rate between coordinates q_i and q_j [2, 6]:

$$\mu_{ij} = \frac{\partial F}{\partial q_i} / \frac{\partial F}{\partial q_j}, \; i, j = 1, ..., \; n, \qquad (1.11)$$

relates components of $\sigma(q)$, coordinates of vector q and substitution rates μ_{ij}. Moreover, points on the surface $F(q) = const$, which satisfy the condition

$$v_i(q) / v_j(q) = \sigma_i(q) / \sigma_j(q) = 1, \qquad (1.12)$$

are characterized by stable proportions.

At the end of this part of the article, the author shows that production function can be expressed in terms of the sensitivity $\sigma(q)$ in the form of an exponential function. The formula for total differential

$$dF(q) = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j} dq_j = \sum_{j=1}^{n} q_j \frac{\partial F}{\partial q_j} \frac{dq_j}{q_j} = \sum_{j=1}^{n} \sigma_j(q) F(q) \frac{dq_j}{q_j}, \qquad (1.13)$$

should be represented in the form

$$\frac{dF(q)}{F(q)} = d \ln F(q) = \sum_{j=1}^{n} \sigma_j(q) d \ln q_j, \qquad (1.14)$$

And then there is a multiplicative form for production function:

$$F(q) = F_0 \prod_{j=1}^{n} \exp(\phi_j(q)), \qquad (1.15)$$

where $\phi_j(q) = \int \sigma_j(q) d \ln q_j, \; j = 1, ..., \; n,$ F_0 – constant.

The second part of the article is devoted to the optimality with respect to sensitivity and return of resources. Optimization of resources is the key task of modern firm management. Two classical problems of the theory of the firm are devoted to this issue. The first one (the task of the theory of demand) is related to the acquisition of a set of resources $(q_1, ..., \; q_n)$ at their market prices $p_1, ..., \; p_n$ respectively, which satisfies the budget constraint $p_1 q_1 + p_2 q_2 +, ..., + p_n q_n \leq I$ and maximizes the utility function of the consumer $u(q_1, ..., \; q_n)$. It is assumed that this function is concave ascending series on the set of benefits $Q \subset E_n^+$ and has continuous partial derivatives of the first and second order satisfying the conditions $\partial u / \partial q_j > 0, \; \partial^2 u / \partial q^2 < 0, \; i, j = 1, ..., \; n$ [6, 13]. It is represented as a mathematical programming task [2, 6].

$$\begin{aligned} &u(q) \rightarrow \max. \\ &pTq \leq I, q \geq o \end{aligned} \qquad (2.1)$$

The second task is related to the minimizing of production factors costs with fixed production output $F(q_1, ..., q_n) = \bar{y}$. If cost function is determined as $r(q) = p^T q$, where $p = (p_1, ..., p_n)^T$, $q = (q_1, ..., q_n)^T$, this task can be represented as a mathematical programming task

$$r(q) = p^T q \rightarrow \min_{F(q)=\bar{y}}. \qquad (2.2)$$

After calculations (See 2.3. and 2.4.) the author concludes that costs are proportional to resources return $v(q^*) = \sigma(q^*) F(q^*),$. The formula $F(q^*) / r(q^*) = \bar{y} / r(q^*) = 1 / \lambda \sigma(q^*)$ determines productivity of a firm. The formula expression for the marginal rate of technical substitution of factors is the following

$$\mu_{ij} = \frac{\partial F(q^*)}{\partial q_i} / \frac{\partial F(q^*)}{\partial q_j} = q_i^* \sigma_i(q^*) / q_j^* \sigma_j(q^*), \; i, j = 1, ..., \; n. \qquad (2.5)$$

So, the decision q^* is optimal in relation to the costs of economic factors. Optimality problem of resources relative to their return comes down to solving the maximization of efficiency or return of resources $v(q) = \nabla F(q)^T q = \sigma(q) F(q)$ on the surface $F(q) = \bar{y}$, under the assumption that the figure $\sigma(q)$ decreases monotonically. Mathematical programming task will have the form

$$v(q) = \sum_{j=1}^{n} \frac{\partial F(q)}{\partial q_j} q_j \rightarrow \max_{F(q)=\bar{y}}. \qquad (2.6)$$

After certain calculations (See 2.7. – 2.13.) the author gives the modified expression for the marginal rate of technical substitution of factors

$$\mu_{ij} = \frac{\partial F(q^v)}{\partial q_i} / \frac{\partial F(q^v)}{\partial q_j} = (Hq^v)_i / (Hq^*)_j, \; i, j = 1, ..., \; n. \qquad (2.14)$$

Relative position of q^* and q^v in case of two production factors are shown on Pic. 3. At the point q^* the level of costs is $r(q^*) = \lambda \sigma(q^*) \bar{y}$ and is less than the level $r(q^v)$ corresponding to the point q^v, but at the point q^v there is the most dynamic return of factors $v(q^v)$, the greatest (relative) sensitivity of relation «input – output» $\sigma(q^v)$ and, as well as the agreed level of interaction of factors $\theta(q^v) = -q^{vT} Hq^v$.

Finally, it should be noted that all points of level lines $F(q) = \bar{y}$ that connect solutions q^* and q^v essentially constitute a «reasonable compromise» between the criteria: the minimum cost function r (q) = $p^T q$ and maximum of functions v(q) = F(q)Tq and (q).

In the theory of production functions two mathematical scheme (or models) that assume a constant figure of the sensitivity function σ (q) have obtained wide application. One of them is the abovementioned production function of Cobb – Douglas in the form

$$F(q) = a_0 q_1^{\alpha_1} q_2^{\alpha_2} ... q_n^{\alpha_n} \qquad (3.1)$$

where $\alpha_j \in (0, 1), \; j = 1, ..., n,$ – constant parameters.

The second model is the so –called production function with constant malleability

$$F(q) = \alpha_0 (\alpha_1 q_1^{-p} + ... + \alpha_n q_n^{-p})^{-m/p}, \alpha_0, m > 0, p \geq -1, \qquad (3.2)$$

where there is a correlation σ (q) = m. In the work [15] the possibility of production function approximation was discussed, in the following form

$$F(q) = F_0 \prod_{i=1}^{n} (1 - e^{-\alpha_i q_i}), \qquad (3.3)$$

where σ (q) is evaluated at all isoquants of the function F (q) and is a decreasing function.

Conclusions.

Thus, in the case of a quadratic approximation of the production function point $q^v = q^F / 2$ for which there is a condition λ = 0, is the most appropriate in terms of both characteristics v(q) and σ(q) . At this same point, the measure of sensitivity is 2/3, the function $F(q^v)$ is 3/4 of its maximum, and the effective return function $v(q^v)$ is 2/3 size of $F(q^v)$.

There are a number of important applications in which the non-linearity and losses of sensitivity and the decreasing conversion efficiency of resources in the market value are the dominant characteristic. These challenges include: the nonlinear accelerator linking induced volume of capital investments with the change in output of the economic system [16]; macroeconomic dynamics model «CVP», describing the dependence of cash receipts and expenditures of resources on production output [17]; dependence of performance on technological productivity, labor productivity and management productivity [11, 18]; spatio – temporal dynamics of the value function or utility for decision-making that characterizes the behavior of economic agents in a situation of choice [2, 19] and many others.

These phenomena and effects are similar to the role and influence of the nonlinear nature of the «central causation», which underlies the structure and activity of the respective systems. H. Simon said that «metaphor and analogy can be very useful, but they can also take away from the truth. All depends on how essential or surface is the similarity, grasped in the metaphor» [20].

Nonlinear effects of spatio – temporary nature are present in natural and artificial systems [21]. They appear with remarkable consistency in engineering, economics and organizational management in the context of high-performance systems, and their role and influence are discussed with growing intensity in the scientific works of Russian and foreign authors.

Keywords: system, nonlinearity, production functions, costs, efficiency, sensitivity theory, theory of the firm, decreasing efficiency, resources, optimization.

REFERENCES

1. Klir G. Systemology. Experience in system tasks solution [Russian title: *Sistemologiya. Opyt resheniya sistemnyh zadach. Per. s angl.*] Moscow, Radio i svyaz' publ., 1990, 544 p.
2. Hall A. D. A Methodology for Systems Engineering [Russian title: Opyt metodologii dlya sistemotehniki]. Moscow, Sovetskoe radio publ., 1975, 607 p.
3. Cohen, Avi J., Harcourt G. C. Whatever Happened to the Cambridge Capital Theory Controversies? [Russian title: *Sud'ba diskussii dvuh Kembridzhey o teorii kapitala*]. *Voprosy ekonomiki*, 2009, Iss. 8, pp. 4–27.
4. Kleyner, G. B. Production functions: theory, methods, application [Russian title: *Proizvodstvennye funktsii: teoriya, metody, primenenie*]. Moscow, Finansy i statistika publ., 1986, 239 p.
5. Mankiw, N. G. Macroeconomist as a scientist and an engineer [Russian title: *Makroekonomist kak uchenyj i inzhener*]. *Voprosy ekonomiki*, 2009, Iss. 8, pp. 86–103.
6. Intriligator, Michael D. Mathematical Optimization and Economic Theory [Russian title: *Matematicheskie metody optimizatsii i ekonomicheskaya teoriya*]. Moscow, Ayris Press publ., 2002, 564 p.
7. Staveren, I. The Ethics of Efficiency [Russian title: *Etika effektivnosti*]. *Voprosy ekonomiki,* 2009, Iss. 12, pp. 58–71.
8. Bazaraa, Mokhtar S., Sherali, Hanif D., Shetty, C. M. Nonlinear Programming: Theory and Algorithms. [Russian title: *Nelineynoe programmirovanie. Teoriya i algoritmy*]. Moscow, Mir publ., 1982, 580 p.
9. Aubin, Jean-Pierre. Non-linear analysis and its economic usage [Russian title: *Nelineynyj analiz i ee ekonomicheskie prilozheniya*]. Moscow, Mir publ., 1988, 264 p.
10. Sage, Andrew P., White, Chelsea C. Optimum Systems Control [Russian title: *Optimal'noe upravlenie sistemami*]. Moscow, Radio i svyaz' publ., 1982, 392 p.
11. Daft, Richard L. Organization Theory and Design [Russian title: *Teoriya organizatsii*]. Moscow, YuNITI-DANA publ., 2006, 736 p.
12. Demidenko, E. Z. Dinamization method of regression coefficients. Automatics and teleautomatics [*Metod dinamizatsii koeffitsientov regressii*]. *Avtomatika i telemehagnika*, 1987, № 9, pp. 77–83.
13. Ekeland, Ivar. Elements of mathematical economics [Russian title: *Elementy matematicheskoy ekonomiki*]. Moscow, Mir publ., 1983, 245 p.
14. Campbell, Andrew, Luchs, Kathleen Sommers. Strategic Synergy [Russian title: *Strategicheskiy sinergizm*]. St. Petersburg, Piter publ., 2004, 416 p.
15. Sarkisyan, R. E. Preferences, utility and measures of sensitivity of multicriterial alternatives [*Predpochteniya, poleznost» i mery chuvstvitel'nosti mnogokriterial'nyh al'ternativ*]. *Ekonomika i matematicheskie metody*, 2009, Vol. 45, Iss. 3, pp. 87–105.
16. Allen, Roy George Douglas. Mathematical Analysis for Economists [Russian title: *Matematicheskaya ekonomika: Per. s angl. ed. by. Vaynshteyn, A.L.*] Moscow, Mir publ., 1963, 580 p.
17. Drury, C. Management and production accounting: an introductory course: a textbook for university students [Russian title: *Upravlencheskiy i proizvodstvennyj uchet. Vvodnyj kurs: textbook*]. Moscow, YuNITI-DANA publ., 2005, 735 p.
18. Management consulting. Professional basics. [*Upravlencheskoe konsul'tirovanie. Vvedenie v professiyu*]. Ed. by Kubra, M. Moscow, Planum publ., 2004, 976 p.
19. DellaVigna, Stefano. Psychology and Economics: Evidence from the Field [Russian title for journal publication: *Psihologiya i ekonomika: Rezul'taty empiricheskih issledovaniy*]. Part 1: non-typical preferences [*Chast I: Nestandartnye predpochteniya*]. *Voprosy ekonomiki*, 2011, Iss. 4, pp. 50–77.
20. Simon, Herbert A. Theories of Decision-Making in Economics and Behavioral Science. Firm Theory [Russian title: *Teoriya prinyatiya resheniy v ekonomicheskoy teorii i nauke o povedenii. Teoriya firmy*]. St. Petersburg, 1995, pp. 54–72.
21. Mainzer, Klaus. Thinking in Complexity: The Computational Dynamics of Matter, Mind and Mankind. [Russian title: *Slozhnosistemnoe myshlenie: Materiya, razum, chelovechestvo. Novyj sintez*]. Moscow, Knizhnyj dom «Librokom» publ., 2009, 464 p.

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