Adequacy of the Gravity Model of Railway Passenger Flows

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ABSTRACT

The most accurate modelling of spatial distribution of passenger flows is a prerequisite for successful planning of development of the transport system. It is the basis for calculation of a predictive trip matrix. An approach based on the gravity model is among main modelling methods.

The work investigates the issue of the adequacy of the gravity model with a double constraint and an exponential-power function of gravitation. It is this specification of the model and its particular cases with exponential and power functions of gravitation that are most often used to estimate spatial distribution of passenger flows both in theoretical and applied research.

Calibration and validation of the specified model is shown on the observed (actual) matrix of railway passenger origin-destination matrix. It was built with the help of the data of Express [railway ticketing] ADB ACS: the number of tickets sold for long-distance trains for all the pairs of directly linked stations.

Since calibration of the gravity model can be carried out by different methods (depending on how the model incorporates stochasticity, which is responsible for differences between the modelled and observed data), after a detailed analysis of the most common methods for calibrating the gravity model, the approach was chosen based on the maximum likelihood estimation. The work also analyses the gravity model validation tools used to estimate the proximity between the observed and modelled trip matrices.

Comparison of the modelled and observed trip matrices resulted in the conclusion that the gravity model under consideration predicts several aggregate indicators with a high degree of accuracy: total passenger turnover, average travel distance, and travel distance distribution. At the same time, it is shown that the error in the forecast of passenger flow for most individual origin-destination trips is quite large. This circumstance significantly reduces the possibility of practical application of the gravity model for the analysis and modelling of passenger flows in long-distance railway passenger traffic.

Keywords: gravity model, railway passenger traffic, trip matrix, model calibration, model validation, model adequacy.

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INTRODUCTION

In Russia, the overall structure of intercity passenger transportation is dominated by road and railway transport. The main volume of passenger transportation with a length of up to 300 km falls on bus and suburban railway routes. However, long-distance rail traffic plays the main role for trips over distances from 300 to 1000 km (for longer routes, air traffic is of primary importance). It is railway transport that to a large extent ensures connectivity (including social one) of the territory of the Russian Federation and interregional transport accessibility, improvement of which is designated as a priority goal for development of the transport system[1].

A necessary prerequisite for successful planning of development of the transport system is associated with the most accurate modelling of spatial distribution of passenger flows, on the basis of which the forecast trip matrix (TM) is calculated. Two main methods used for modelling are entropy and gravity ones. The former is based on the assumption that the transport system, considered as a system of movement of individuals, is closed and tends to reach an equilibrium state corresponding to the maximum entropy[2]. A detailed presentation of this method can be found, for example, in the monograph [1].

The latter method is based on an analogy with Newton's law of gravity. With this approach, it is assumed that the passenger flow depends on the size of the points of departure and arrival and the distance between them, and that between near and large points, the passenger flow is greater than between small and distant ones. The specific form of such dependence is determined by the modelled system and additional constraints.

The first attempts to use the gravity model to describe spatial distribution of various types of economic and social interaction were made in the 1930s [2], and possibly even earlier [3]. However, this model became widespread in the 1950s [4] and has since been actively used to model transport, trade, financial, migration flows, etc. (examples of application of gravity model in fields not related to transport along with the relevant references can be found in the article of one of the authors [5]).

In the 1960s and 1970s, the mathematical properties of the gravity model were studied in detail by the efforts of several scientists. In particular, the works [6–9] proposed and rigorously substantiated methods for calibrating the model based on empirical data. A rather complete exposition of these and other theoretical issues is contained in the review article [3] and the monograph [10]. Also, during this period, the application of the gravity model began to predict the spatial distribution of passenger flows in specific cities and territories when developing plans for development of transport infrastructure [11]. However, during those years, the practical use of the gravity model was rather limited, since the collection of empirical data necessary for calibrating the model was carried out through surveys and direct observations. This required the involvement of a large number of traffic checkers and, accordingly, implied significant costs.

The situation concerning availability of data has begun to change since the late 1990s due to development of information technology. The widespread introduction of video surveillance systems on the roads, the use of cashless payments, the emergence of smartphones with geolocation and other similar technologies have made it possible to obtain sufficiently complete and reliable information about transport flows.

As availability of such data has increased, the number of works aimed at testing the adequacy of the gravity model for modelling the spatial distribution of passenger flows for various territories and transport systems has increased significantly. Results of this kind can be found in [12–19] and many other works. In particular, it was shown in [12] that the gravity model quite well describes the spatial distribution of air passenger transportation volume across the countries of Central Europe.

The article [13] used data on daily work trips in six countries to test the adequacy of various modifications of the gravity model. Among other things, this work showed that for such data, the gravity model gives a sufficiently good forecast for small and medium distances, but for large distances, the accuracy of the forecast is significantly reduced. Data on work trips were also used in [14] to assess the possibility of using

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[1] Transport Strategy of the Russian Federation till 2030 with a forecast for the period up to 2035. Approved by the order of the Government of the Russian Federation of 27.11.2021 No. 3363-r. (in Russian). [Electronic resource]: http://static.government.ru/media/files/7enYF2uL5kFZlOOpQhLl0nUT91RjCbeR.pdf. Last accessed 22.12.2022.

[2] Gasnikov, A. V., Klyonov, S. L., Nurminsky, E. A. [*et al*]. Introduction to Mathematical Modelling of Transport Flows: Study guide. Ed. by A. V. Gasnikov. Moscow, MFTI publ., 2010, 362 p. ISBN 978-5-7417-0334-2.

a gravity model, calibrated using the data of one region, to predict trips in another region. In [15], the gravity model was validated based on data on carpooling trips made in one of the regions of Russia.

In [16], the gravity model was tested for cargo railway flows between provinces in China. The article [19] contained a similar study regarding passenger transportation by rail. In this work, based on data on the volume of passenger transportation on one of the lines of the Spanish railways, the authors showed that the gravity model approximates the observed data much better than the Poisson regression.

Checking the adequacy of the gravity model for specific transport systems, territories and modes of transport is necessary to justify the legitimacy of using the gravity model to solve applied problems. From a practical point of view, the gravity model is an integral part of the four-step transportation model [1], which is actively used in making managerial decisions on development of transport infrastructure [3,4], and is implemented in all modern transport modelling software (PTV Visum, TransNet, and others).

The gravity model can be used in the four-step model at the second step to calculate the total TM (for all modes of transport), after which, at the third step, for each origin-destination segment, the splitting of passenger traffic per different modes of transport is calculated. Also, the gravity model can be used in the form of a so-called synthetic gravity model for simultaneous calculation of TM and splitting per mode of transport (that is, the second and third steps are combined). In this case, the gravity model is used to calculate TM for each mode of transport separately. Accordingly, practically significant results can be obtained only if the gravity model adequately describes the spatial distribution of passenger flows for each mode of transport separately.

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[3] Methodological guidelines for development and implementation of measures for organisation of traffic. Organisation of traffic at controlled intersections. JSC Scientific Research Institute of Automobile Transport, Moscow, 2017. [Electronic resource]: https://e-ecolog.ru/docs/zZ7TAN6o8Pn-TZ7JAQqI2/. Last accessed 22.12.2022.

[4] STO [organisation's standard] AVTODOR 2.2–2013. Design, construction, operation of highways. Recommendations for predicting traffic intensity on toll sections of highways of the Avtodor state company and income from their operation. Moscow, Standard of the Avtodor State Company, 2013. [Electronic resource]: https://studylib.ru/doc/2002300/rekomendacii-po-prognozirovaniyu-intensivnosti-dorozhnogo/. Last accessed 22.12.2022.

In particular, when creating a transport model on a national scale, this also applies to long-distance railway transportation. Therefore, the main *objective* of the study was to test the adequacy of the gravity model for modelling the spatial distribution of passenger flows in long-distance railway transportation.

RESULTS
Gravity Model

Let passengers make trips from points of departure $i = 1, 2,…, I$ to points of destination $j = 1, 2,…, J$ and c_{ij} which is the generalised cost of a trip from point i to point j is known (generalised cost is a value that includes all the time and financial expenses of a passenger for the trip). Let's denote by T_{ij} the number of trips from point i to point j. Obviously, T_{ij} depends on the point of origin i, the point of destination j and the value c_{ij}. One of the simplest forms of such a dependence is a relation of the form:

$$T_{ij} = a_i b_j f\left(c_{ij}\right), \qquad (1)$$

where a_i and b_j are some quantitative characteristics of origin and destination points;

$f(c)$ – a non-negative function that is defined for all $c \geq 0$.

If the function $f(c)$ is decreasing, and the characteristics a_i and b_j are interpreted as the sizes of the points of origin and absorption of passenger flows at the corresponding points, then dependence (1) is a mathematical implementation of the empirical rule «large and near situated objects interact more strongly than small and distant». Due to the similarity with Newton's law of universal gravitation, dependence (1) is called the gravity model.

Let s_i be the number of departures from point i (to all points of arrival), d_j – the number of arrivals at point j (from all points of departure). Then it follows from the obvious identities $\sum_{j=1}^{J} T_{ij} = s_i$, $\sum_{i=1}^{I} T_{ij} = d_j$ that a_i and b_j must satisfy the equalities

$$\begin{cases} \sum_{j=1}^{J} a_i b_j f\left(c_{ij}\right) = s_i \; \forall i = 1,..,I, \\ \sum_{i=1}^{I} a_i b_j f\left(c_{ij}\right) = d_j \; \forall j = 1,..,J. \end{cases} \qquad (2), (3)$$

For given s_i and d_j, the system (2), (3) is a system of equations for the unknowns a_i and b_j. If for any c_{ij} the condition $f\left(c_{ij}\right) > 0$ is satisfied and

$$\sum_{i=1}^{I} s_i = \sum_{j=1}^{J} d_j,$$

then the system (2), (3) has a solution [7]. In this case, the solution will not be unique (if a_i^*, b_j^* is

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a solution to system (2), (3), then θa_i^*, $\theta^{-1}b_j^*$ will also be a solution for any $\theta > 0$). However, it was shown in [7] that for any two different solutions a_i^1, b_j^1 and a_i^2, b_j^2, the equality $a_i^1 b_j^1 = a_i^2 b_j^2$ will hold for any i, j.

This means that T_{ij} does not depend on which particular solution of system (2), (3) will be used in (1). Thus, the gravity model (1) establishes a one-to-one correspondence between the TM T_{ij} on the one hand and a pair of vectors ($a_1, \cdots, a_I$), ($b_1, \cdots, b_J$) on the other.

The function $f(c_{ij})$ is called the gravitation function. It shows the «readiness» of the passenger to make a trip, depending on the size of the generalised cost c_{ij}. As a function of gravitation $f(c_{ij})$, the researchers often use a power function $f(c_{ij}) = c_{ij}^{-\mu}$, an exponential function $f(c_{ij}) = \exp(-\mu c_{ij})$ or a combined function $f(c_{ij}) = c_{ij}^{\gamma} \exp(-\mu c_{ij})$, which is also called exponential-power function.

The power and exponential functions of gravitation monotonically decrease for $c_{ij} > 0$, so for these functions, the lower is the generalised cost c_{ij}, the greater is the «readiness» of a passenger to make a trip. In many cases, this is true, but in some situations, for small values of c_{ij}, this may not be the case. For example, for intra-city public transport, many passengers have a lower «willingness» to travel between adjacent stops than for a longer distance (between two adjacent stops, such passengers prefer to walk). A similar situation will be true for long-distance railway transportation: for trips over short distances, passengers choose commuter trains or buses. Consequently, in such cases, to describe gravity, the combined function is more adequate, which, in the case of $\gamma > 0$, first increases and then decreases. Therefore, further we will consider (1) with the combined gravitation function, that is, in the form:

$$T_{ij} = a_i b_j c_{ij}^{\gamma} \exp(-\mu c_{ij}). \qquad (4)$$

Calibration of the Gravity Model using the Observed TM

Let us assume that as a result of observations, the TM $N = \{n_{ij}\}$, $i = 1, \ldots, I$, $j = 1, \ldots, J$ is obtained and consider the problem of estimating the parameters a_i, b_j, γ and μ of the model (4) using the observed matrix N. Obviously, the

method of estimation depends on how exactly the model incorporates the stochasticity that is the cause of the deviation of the observed values n_{ij} from the modelled values T_{ij}. Let's take a closer look at some of the most popular approaches.

1. The simplest option is to assume that $n_{ij} = T_{ij} + \varepsilon_{ij}$ where ε_{ij} is a random variable. In this case, it is natural to use the least squares method (LSM) to estimate the model parameters. As a result, we get the minimisation problem:

$$S(a_i, b_j, \gamma, \mu) = \sum_{i,j} (n_{ij} - T_{ij})^2 =$$
$$= \sum_{i,j} (n_{ij} - a_i b_j c_{ij}^{\gamma} \exp(-\mu c_{ij}))^2 \xrightarrow{a_i, b_j, \gamma, \mu} min,$$

which, as a result of applying standard methods of mathematical analysis, leads to a system of equations for determining the parameters a_i, b_j, γ and μ:

$$\begin{cases} \dfrac{\partial S}{\partial a_i} = 0 \, \forall i = 1,..,I, \\ \dfrac{\partial S}{\partial b_j} = 0 \, \forall j = 1,..,J, \\ \dfrac{\partial S}{\partial \mu} = 0, \\ \dfrac{\partial S}{\partial \gamma} = 0. \end{cases} \Leftrightarrow \begin{cases} \sum_i T_{ij}(n_{ij} - T_{ij}) = 0 \, \forall i = 1,..,I, \\ \sum_j T_{ij}(n_{ij} - T_{ij}) = 0 \, \forall j = 1,..,J, \\ \sum_{i,j} C_{ij} T_{ij}(n_{ij} - T_{ij}) = 0, \\ \sum_{i,j} \ln C_{ij} T_{ij}(n_{ij} - T_{ij}) = 0. \end{cases} \quad (5)$$

The weak point of this approach is that the use of least squares requires restrictions on ε_{ij}, which in this case, most likely, will not be satisfied. In particular, ε_{ij} will not be homoscedastic (obviously, the variance ε_{ij} depends on the value of T_{ij}). Also, a significant disadvantage of using LSM is the lack of a meaningful interpretation for equations (5).

2. Another approach, which was first rigorously stated in [8], is based on the assumption that each passenger randomly and independently of other passengers chooses one origin-destination segment (i, j) from $I \cdot J$ correspondences to make a trip. Moreover, the probability p_{ij} of choosing trip (i, j) is the same for all passengers. With this approach, T_{ij} is interpreted as the theoretical selection frequency of trip (i, j) with $n = \sum_{i,j} n_{ij}$ passengers, that is, $T_{ij} = n p_{ij}$. Therefore, the gravity model (4) can be written as:

$$p_{ij} = \frac{1}{n} a_i b_j c_{ij}^{\gamma} \exp(-\mu c_{ij}). \qquad (6)$$

To estimate the parameters $a_1, \ldots, a_I, b_1, \ldots, b_J, \gamma, \mu$ of the model (6) based on the observed matrix N, the maximum likelihood method is used. In this case, the likelihood function will look like:

$$L = L(a_i, b_j, \gamma, \mu) = \prod_{i=1}^{I}\prod_{j=1}^{J}(p_{ij})^{n_{ij}} =$$
$$= \prod_{i=1}^{I}\prod_{j=1}^{J}\left(n^{-1}a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij})\right)^{n_{ij}}.$$

Taking the logarithm of the function L and considering that $n_{ij}\ln n^{-1}$ does not depend on the model parameters, and also that the probabilities p_{ij} must satisfy the normalisation condition $\Sigma p_{ij} = 1$, we arrive at the conditional optimisation problem:

$$\ln L = \sum_{i=1}^{I}\sum_{j=1}^{J}n_{ij}\left(ln a_i + \ln b_j + \gamma \ln c_{ij} - \mu c_{ij}\right) \xrightarrow{a_i, b_j, \gamma, \mu} max,$$
$$\sum_{i=1}^{I}\sum_{j=1}^{J}a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij}) = n.$$

Next, we use the method of Lagrange multipliers. The Lagrange function will look like:
$$F(a_i, b_j, \gamma, \mu, \lambda) = \sum_{i=1}^{I}\sum_{j=1}^{J}n_{ij}\left(ln a_i + \ln b_j + \gamma \ln c_{ij} - \mu c_{ij}\right) -$$
$$-\lambda\left(\sum_{i=1}^{I}\sum_{j=1}^{J}a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij}) - n\right).$$

Equating the partial derivatives of the function F to zero, we get (7)–(11).

If each of equations (7) is multiplied by a_i and summed over i (or each of equations (8) is multiplied by b_j and summed over j), then we get:
$$\lambda\sum_{i=1}^{I}\sum_{j=1}^{J}a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij}) = \sum_{i=1}^{I}\sum_{j=1}^{J}n_{ij} = n. \qquad (12)$$

From here and from (11) it follows that $\lambda = 1$. Also (12) means that equation (11) is a consequence of each of the group of equations (7) and (8). Therefore, taking into account (4), the system of equations (7)–(11) takes the form:

$$\begin{cases} \sum_{j=1}^{J}T_{ij} = \sum_{j=1}^{J}n_{ij}\ \forall i = 1,..,I, \\ \sum_{i=1}^{I}T_{ij} = \sum_{i=1}^{I}n_{ij}\ \forall j = 1,..,J, \\ \sum_{i=1}^{I}\sum_{j=1}^{J}T_{ij}c_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J}n_{ij}c_{ij}, \\ \sum_{i=1}^{I}\sum_{j=1}^{J}T_{ij}\ln c_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J}n_{ij}\ln c_{ij}. \end{cases} \qquad (13)-(16)$$

Thus, estimates of the parameters $a_1,..., a_I$, $b_1,..., b_J$, γ, μ a_I can be found from the system of equations (13)–(16). In contrast to system (5), here all equations have a transparent interpretation, which corresponds well to the content of the gravity model. In particular, equation (13) means that for each point i, the model number of departures (to all points of arrival) must match the observed number of destinations. Similarly, equation (14) means that the corresponding modelled and observed arrivals coincide. Equation (15) shows that the total modelled cost of all trips should equal the total observed cost of all the trips. If both parts of (15) are divided by $\sum_{i,j}T_{ij} = \sum_{i,j}n_{ij}$, then it can be interpreted as an equality of the corresponding average costs. The last equation has the same meaning as (15), but only for the logarithmic generalised cost c_{ij}.

3. Another approach was proposed in [9] to include stochasticity in the gravity model. In this work, the number of trips from point i to point j is a discrete random variable Θ_{ij}, and the value T_{ij} given by formula (4) is interpreted as the mathematical expectation of the value Θ_{ij}, that is:
$$M[\Theta_{ij}] = T_{ij} = a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij}).$$

Let's assume that the values Θ_{ij} ($i=1,...,I$, $j=1,...,J$) are independent in the aggregate and obey the same distribution law (with different parameter values):
$$Pr(\Theta_{ij} = k) = \varphi(k, T_{ij}) = \varphi(k, a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij})).$$

Then, to estimate the parameters $a_1,..., a_I$, $b_1,..., b_J$, γ, μ based on the observed matrix N, we use the maximum likelihood method. In this case, the likelihood function will look like:

$$L = L(a_i, b_j, \gamma, \mu) = \prod_{i=1}^{I}\prod_{j=1}^{J}Pr(\Theta_{ij} = n_{ij}) =$$
$$= \prod_{i=1}^{I}\prod_{j=1}^{J}\varphi(n_{ij}, a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij})).$$

After taking the logarithm of the function L, we obtain the following optimisation problem:

$$\begin{cases} \dfrac{\partial F}{\partial a_i} = \dfrac{1}{a_i}\sum_{j=1}^{J}n_{ij} - \lambda\sum_{j=1}^{J}b_j c_{ij}^{\gamma}\exp(-\mu c_{ij}) = 0\ \forall i = 1,...,I, \\ \dfrac{\partial F}{\partial b_j} = \dfrac{1}{b_j}\sum_{i=1}^{I}n_{ij} - \lambda\sum_{i=1}^{I}a_i c_{ij}^{\gamma}\exp(-\mu c_{ij}) = 0\ \forall j = 1,...,J, \\ \dfrac{\partial F}{\partial \mu} = -\sum_{i=1}^{I}\sum_{j=1}^{J}n_{ij}c_{ij} - \lambda\sum_{i=1}^{I}\sum_{j=1}^{J}a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij})(-c_{ij}) = 0, \\ \dfrac{\partial F}{\partial \gamma} = \sum_{i=1}^{I}\sum_{j=1}^{J}n_{ij}\ln c_{ij} - \lambda\sum_{i=1}^{I}\sum_{j=1}^{J}a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij})\ln c_{ij} = 0, \\ \dfrac{\partial F}{\partial \lambda} = \sum_{i=1}^{I}\sum_{j=1}^{J}a_i b_j c_{ij}^{\gamma}\exp(-\mu c_{ij}) - n = 0. \end{cases} \qquad (7)-(11)$$

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$$\begin{cases} \dfrac{\partial \ln L}{\partial a_i} = \dfrac{1}{a_i} \sum_{j=1}^{J} T_{ij} \dfrac{d \ln \varphi}{d T_{ij}} = 0 \ \forall i = 1,\dots,I, \\[2mm] \dfrac{\partial \ln L}{\partial b_j} = \dfrac{1}{b_j} \sum_{1=1}^{I} T_{ij} \dfrac{d \ln \varphi}{d T_{ij}} = 0 \ \forall j = 1,\dots,J, \\[2mm] \dfrac{\partial \ln L}{\partial \mu} = -\sum_{i=1}^{I} \sum_{j=1}^{J} T_{ij} \dfrac{d \ln \varphi}{d T_{ij}} c_{ij} = 0, \\[2mm] \dfrac{\partial \ln L}{\partial \gamma} = \sum_{i=1}^{I} \sum_{j=1}^{J} T_{ij} \dfrac{d \ln \varphi}{d T_{ij}} \ln c_{ij} = 0. \end{cases} \qquad (17)\text{–}(20)$$

$$\ln L = \sum_{i=1}^{I} \sum_{j=1}^{J} \ln \varphi \left(n_{ij}, T_{ij} \right) =$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} \ln \varphi \left(n_{ij}, a_i b_j c_{ij}^{\gamma} \exp\left(-\mu c_{ij}\right) \right) \xrightarrow{a_i, b_j, \gamma, \mu} max.$$

Equating the partial derivatives of the function $\ln L$ to zero, we obtain a system of equations for determining the required parameter estimates (17)–(20).

The specific form of equations (17)–(20) depends on the choice of distribution φ. Quite often, the Poisson distribution is used as φ:

$$\varphi\left(T_{ij}, n_{ij}\right) = \frac{\exp\left(-T_{ij}\right) T_{ij}^{n_{ij}}}{n_{ij}!}.$$

It can be verified by direct calculations that in this case system (17)–(20) turns into a system of equations (13)–(16). If we take the normal distribution as φ:

$$\varphi\left(T_{ij}, n_{ij}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\left(T_{ij} - n_{ij}\right)^2}{2\sigma^2}\right),$$

then system (17)–(20) turns into system of equations (5). Moreover, this will be true for any standard deviation σ.

Thus, the third approach to modelling the stochasticity of the gravity model generalises the first and second approaches in the sense that it leads to identical systems of equations for determining model parameter estimates. At the same time, it should be emphasised that the first and third approaches model stochasticity in virtually the same way: in the first approach, $M\left[T_{ij}\right] = M\left[T_{ij} + \varepsilon_{ij}\right]$ under a fairly natural assumption that $M\left[\varepsilon_{ij}\right] = 0$, that is, in both approaches, the modelled number of origin-destination trips is interpreted as the mathematical expectation of the observed number of the trips. However, the second approach is fundamentally different from them: in it, the modelled relative frequency of origin-destination trips is the probability that a passenger chooses this trip.

As mentioned above, the first approach and equations (5) obtained on its basis are rarely used to estimate the parameters of the gravity model and mainly in theoretical studies (one example of using the first approach to estimate the parameters of the TM based on the entropy model can be found in [20]), which is due to the need to impose not very realistic restrictions on ε_{ij}. At the same time, equations (13)–(16) are the main tool for estimating the parameters of the gravity model both in scientific research and when creating transport models for specific areas. Therefore, in this work, to calibrate model (4), we will use the system of equations (13)–(16). This system does not have an analytical solution, so it is necessary to use approximate methods to solve it.

Equations (13)–(14) actually represent system (2)–(3), where the observed numbers of departures and arrivals are used as s_i and d_j, which means that the above statements about the existence and uniqueness of the solution will be valid for them (this again emphasises the naturalness of using system (13)–(16) for model calibration). Therefore, for given values of γ and μ, system (13)–(14) uniquely determines all products $a_i b_j$, that can be found approximately using the method of simple iterations. To do this, a_i are expressed from equations (13), and b_j are expressed from equations (14). Then all a_i are given some initial values (usually one) and b_j are calculated from them based on equations (14). Based on the obtained values of b_j, a_i are calculated from equations (14) and so on. The process is continued until equalities (13)–(14) are fulfilled with a given level of accuracy. Such an algorithm is called the Fratar method, or the Furness method, or the biproportional algorithm (for details, see the monograph [21]).

Thus, equations (13)–(14) define the function $q(\gamma,\mu) = a_i(\gamma,\mu) b_j(\gamma,\mu)$. Therefore, (15)–(16) are equations for γ and μ (21)–(22).

Although the function $q(\gamma,\mu)$ is smooth, it cannot be specified analytically, moreover, only its approximate values are available to us. Therefore, to solve system (21)–(22), it is

Table 1

Main characteristics of observed TM [calculated by the authors]

Characteristics	LTM	STM
Number of cities	98	38
Number of specific origin-destination trips	9604	1444
Number of passengers	3743252	2240018
Number of zero specific origin-destination trips	6217	616
Number of specific origin-destination trips with less than 10 passengers	6504	622
Maximum number of passengers for one specific origin-destination trip	30006	30006
Average number of passengers for all origin-destination trips	390	1551
Average number of passengers for all non-zero specific origin-destination trips	1105	2705
Median number of passengers for all specific origin-destination trips	0	134
Median number of passengers for all non-zero specific origin-destination trips	157	835
Passengers turnover, thousand people-km	4563511	3009016
Average trip distance, km	1219,13	1348,3

$$
\begin{cases}
\sum_{i=1}^{I}\sum_{j=1}^{J} T_{ij}(\gamma,\mu)c_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J} q(\gamma,\mu)c_{ij}^{\gamma}\exp(-\mu c_{ij})c_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij}c_{ij}, \\
\sum_{i=1}^{I}\sum_{j=1}^{J} T_{ij}(\gamma,\mu)\ln c_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J} q(\gamma,\mu)c_{ij}^{\gamma}\exp(-\mu c_{ij})\ln c_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J} n_{ij}\ln c_{ij}.
\end{cases}
\tag{21)–(22}
$$

convenient to pass to an equivalent optimisation problem:

$$
H(\gamma,\mu) = H_1(\gamma,\mu) + H_2(\gamma,\mu) =
$$

$$
= \left| \sum_{i,j} T_{ij}(\gamma,\mu)c_{ij} - \sum_{i,j} c_{ij}n_{ij} \right| +
$$

$$
+ \left| \sum_{i,j} T_{ij}(\gamma,\mu)\ln c_{ij} - \sum_{i,j} n_{ij}\ln c_{ij} \right| \rightarrow
\tag{23}
$$

$$
\rightarrow min,
$$

to solve which it is necessary to use gradientless optimisation methods.

Initial Data

To calibrate the gravity model in this work, we used data from the automated control system of the Express ADB ACS, available through the Correspondence and financial results AWP. This system allows getting the number of tickets sold for long-distance trains for any pair of departure and arrival stations, that is, the observed TM.

The data processed covered a month of 2019[5]. The period was typical for prepandemic period. In total, the ACS contains information about almost 37000 stations. But the vast majority of stations generate negligible passenger traffic. For 35000 stations, the number of departed and arrived passengers (for all destinations) is only a few per month. For another 1000 departure (arrival) stations, there are no more than 500 passengers per month, that is, less than 20 people per day. Therefore, it makes no sense to use all

stations to calculate the parameters of the gravity model. It is also necessary to keep in mind that several railway stations can be located in one and the same city, so their passenger flows must be combined, and TM should be considered by stations, but by cities.

To compare the quality of approximation for observed TM of different sizes, we will calibrate the model on two data sets: TM for cities, each of which has a total of more than 20 thousand departures and arrivals, as well as TM for cities, each of which has a total of more than 50 thousand departures and arrivals. For brevity, we will call the first of them a large TM (LTM), and the second a small TM (STM). The statistics of these TM are presented in Table 1.

Note that TM under consideration have a rather large number of zero correspondences (see Table 1). In particular, because there is no direct passenger communication between many cities. The points of departure and arrival within STM, as well as all origin-destination trips with a volume of more than 5000 passengers, are shown in Pic. 1.

Also, visualisation of the observed data for STM case is shown in Pic. 2. Here, a high degree of heterogeneity in distribution of passengers by origin-destination trips is clearly visible: the vast majority of cities have significant passenger flows for a small number of origin-destination trips (from 1 to 4) and only for a few cities distribution of passengers by origin-destination trips looks uniform.
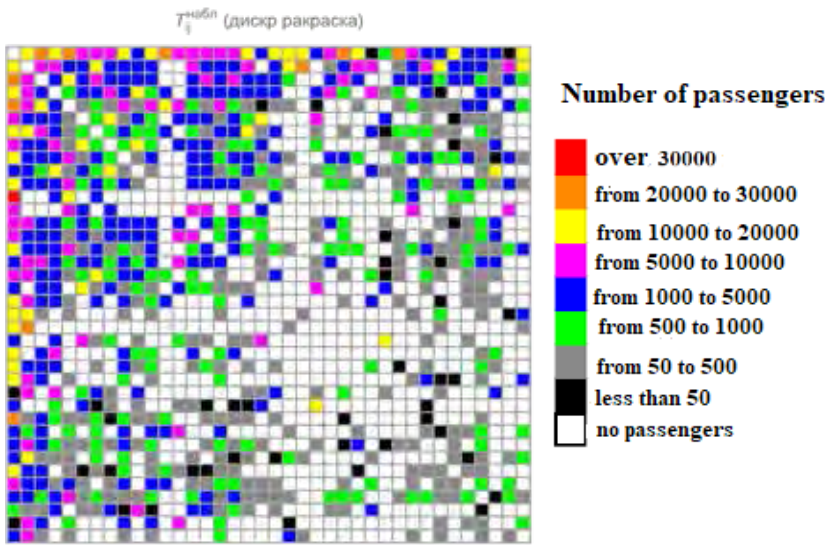
As a generalised cost c_{ij} in the gravity model, we will use the tariff distance between stations.

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[5] Data were collected directly from Express ADB ACS based on the agreement on the access granted in 2019 to the Ural State University of Railway Transport for research purposes.

Pic. 1. STM correspondences with a volume of more than 5000 passengers [built by the authors].

$T_{ij}^{набл}$ (дискр ракраска)

Number of passengers

- over 30000
- from 20000 to 30000
- from 10000 to 20000
- from 5000 to 10000
- from 1000 to 5000
- from 500 to 1000
- from 50 to 500
- less than 50
- no passengers

Pic. 2. Visualisation of STM [built by the authors].

If there is no direct passenger link between stations i and j, then we accept that $c_{ij} = \infty$. We also do the same for the diagonal elements of TM, that is, $c_{ij} = \infty$ for any i. In calculations, a sufficiently large number is used as ∞.

For such a choice of c_{ij}, the left and right sides of equation (21) are the modelled and observed passenger traffic, respectively, and the minimised function $H_1(\gamma,\mu)$ is equal to the absolute deviation of the modelled passenger traffic from the observed one. If both parts of equation (21) are divided by $\sum_{i,j} T_{ij} = \sum_{i,j} n_{ij}$, then we get the model and observed average trip distances (ATD). Equation (22) can be interpreted similarly for a logarithmic distance (logdistance). Thus, the calibration of the gravity model is reduced to the selection of such

values of the parameters γ and μ, at which the modelled passenger turnover coincides with the observed one.

We also note that the parameters γ and μ do not depend linearly on the choice of the measurement unit c_{ij}. In particular, if c_{ij} is measured in kilometers, then at distances of several thousand kilometers characteristic of the Russian railway network, the values of γ and μ may turn out to be less than computer zero. Therefore, the values of c_{ij} will be set in thousand km.

Simulation Results

The Nelder–Mead method was used to minimise function (23). The results obtained are presented in Table 2.

Gravity model calibration results [calculated by the authors]

Value	LTM	STM
Parameter γ	0,3611	0,6616
Parameter μ	1,1515	1,3093
Function to be minimised $H(\gamma,\mu)$	365,1	32,7
Absolute deviation of passenger turnover $H_1(\gamma,\mu)$	199,7	18,29
Absolute passenger turnover deviation for logdistance $H_2(\gamma,\mu)$	165,4	14,44
Relative deviation of passenger turnover $H_1(\gamma,\mu)/\sum c_{ij} n_{ij}$	$4,4\times10^{-5}$	$6,1\times10^{-6}$
Relative deviation of passenger turnover for logdistance $H_2(\gamma,\mu)/\sum n_{ij}\ln c_{ij}$	$5,5\times10^{-4}$	$9,2\times10^{-5}$

As can be seen from the Table 2, in both cases, the model calibration was performed with a sufficiently high accuracy. In particular, the modelled passenger turnover is equal to the observed one with a relative error of less than 0,01 %.

To validate the gravity model, the proximity between the observed and modelled TM is estimated, and the modelled and observed distributions of travel distances are compared (this approach is also available in situations where the observed TM is unknown, and the distribution of travel distances is obtained using surveys). In this case, certain difficulties may arise in the use of standard tools for comparison and interpretation of the results obtained (see [22–24] and, especially, [25]). In particular, the use of the chi-square test to compare TM (and travel distance distributions) almost always leads to rejection of the hypothesis about the coincidence of modelled and observed frequencies, which is due to the sparseness of the observed TM and the very high sensitivity of this criterion for large sample sizes. Similar difficulties arise when using other chi-square criteria.

It is also necessary to use with precaution such indicators as the correlation coefficient, the coefficient of determination, and the like as proximity measures because of strong nonlinearity (for close values of these indicators, the quality of the models can differ significantly [22; 25]). To overcome the difficulties associated with nonlinearity, relative information criteria have been developed that depend on errors «almost linearly» and therefore allow effectively comparing different models [25]. If the task is to check the adequacy of one model, then here it is necessary to proceed from the meaningful formulation of a specific problem and use easily interpretable proximity measures such as the average absolute error and others.

In our case, the distribution of travel distances for the observed and modelled matrices is shown in Pic. 3. Here, the modelled values are in good agreement with the observed ones: the total deviation for all intervals is 14 % for both STM and LTM, that is, only 14 % of the passengers predicted by the model will travel distances different from those observed. In passing, we note that, as mentioned above, the formal application of the chi-square test leads to the conclusion that the modelled and observed frequencies differ: for STM and LTM, the values of the criterion are $\chi^2_{obs}=150031$ and $\chi^2_{obs}=149143$, respectively.

Quantitative estimates of proximity of matrices are presented in Table 3. Here it can be immediately noted that, except for MAE (mean absolute error) and S (standard deviation), all indicators are practically the same for LTM and STM. The strong differences between MAE and S are due to the fact that LTM and STM differ much more in the number of specific origin-destination trips than in the number of passengers.

The obtained values of indicators MAE, MAE_p and S in Table 3 indicate a low level of agreement between the observed and modelled values. In particular, the MAE value is more than 50 % of the average and median number of passengers (see Table 1), and MAE_p shows the same values relative to one passenger. The coefficients of correlation and determination show a higher level of compliance, but as already noted, in this situation this is of secondary importance.

The final conclusions regarding the quality of the obtained model can be drawn from Pic. 4. In the picture, along with the residuals of the model, straight lines $y=0,1x$ and $y=0,5x$ (on a logarithmic scale along the x axis) are plotted, which highlight

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Proximity measures between modelled and observed TM [calculated by authors]

Measure of proximity of matrices	LTM	STM
Mean absolute error $MAE = \sum_{i,j} \left| n_{ij} - T_{ij} \right| / \left(I \cdot J \right)$	242	886
Absolute error averaged per passengers $MAE_p = \sum_{i,j} \left| n_{ij} - T_{ij} \right| / \sum_{i,j} n_{ij}$	0,62	0,57
Standard deviation $S = \left(\sum_{i,j} \left(n_{ij} - T_{ij} \right)^2 / \left(I \cdot J - 1 \right) \right)^{1/2}$	1152	2337
Correlation coefficient $r = cov\left(n_{ij}, T_{ij} \right) / \left(\sigma\left(n_{ij} \right) \sigma\left(T_{ij} \right) \right)$	0,77	0,78
Correlation coefficient r only for $n_{ij} > 0$	0,75	0,75
Determination coefficient $R^2 = 1 - \sum_{i,j} \left(n_{ij} - T_{ij} \right)^2 / \sum_{i,j} \left(n_{ij} - \overline{n_{ij}} \right)^2$	0,55	0,59
Determination coefficient R^2 only for $n_{ij} > 0$	0,51	0,53
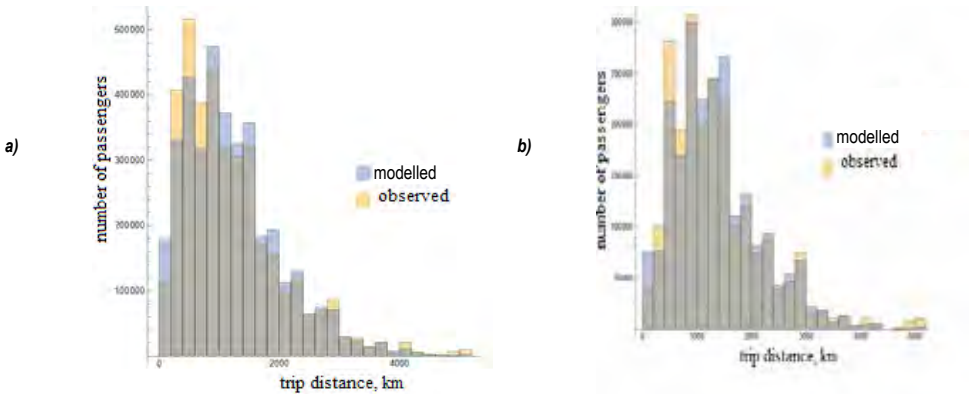
Pic.3. Comparison of the modelled and observed distribution of travel distance for: a) LTM; b) STM [built by the authors].
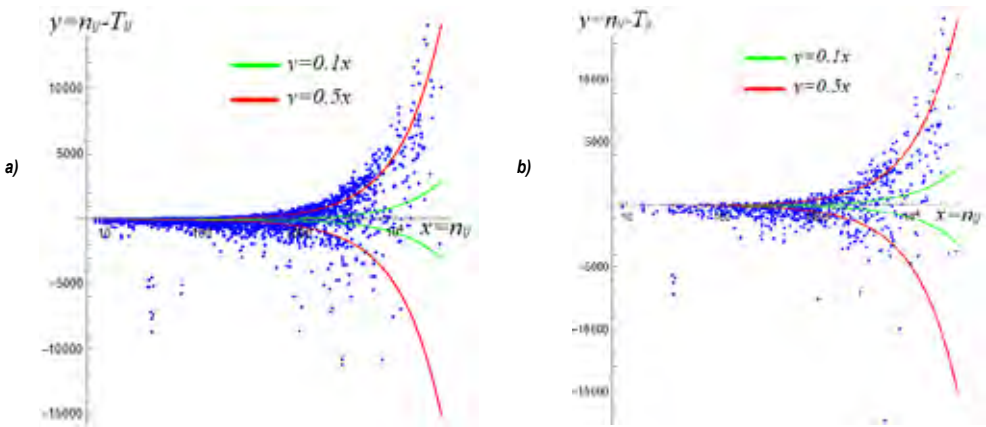
Pic. 4. Model deviations for: a) LTM; b) STM (logarithmic scale is used on the horizontal axis) [built by the authors].

10 % and 50 % of the model deviation for specific origin-destination trips. The Pic. 4 shows that there are few accurate forecasts (with a deviation of less than 10 %), and a lot of inaccurate forecasts (with a deviation of more than 50 %). And this is true for specific origin-destination trips with any volume of passengers. Thus, the gravity model cannot be used to predict passenger flows on individual origin-destination segments.

CONCLUSIONS

Checking the adequacy of the gravitational model with a double constraint and an exponential-power function of gravitation for modelling the

origin-destination trip matrix of a long-distance railway transportation shows that:

1) The model predicts with high accuracy such aggregated characteristics of spatial distribution of passenger flows as the average trip distance, total passenger turnover and trip distance distribution.

2) The accuracy of the model for predicting the passenger flows on individual origin-destination segments is quite low.

Thus, the model can be used to predict the distribution of travel distances, but it cannot be used to predict the volume of individual origin-destination routes.

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