DISTRIBUTION OF STATISTICS IN HYPOTHESES TESTING

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ABSTRACT

The article considers the influence of the form of laws of distribution of single observations on quantile of statistics used for statistical hypotheses testing and having chi-square distribution of or Fisher distribution with the normal distribution of individual observations. Through mathematical modeling on the example of four options it is shown that difference of the distribution law as compared to normal law results in a significant change in the value of quantiles. This is equivalent to an unjustified change in the level of significance in the choice of a decision rule.

Keywords: mathematical statistics, distribution law, quantiles, statistical hypothesis, statistical modeling.

Background. *Testing of statistical hypotheses [1] is characteristic of statistical quality control methods, as well as of methods of identification of errors in measurements [2, 3]. The most commonly used statistics γ have a chi-square distribution and Fisher distribution. In the first case, the statistics may be represented as*

$$\gamma_1 = \chi_N^2 = \sum_{i=1}^{N} \xi_i^2 . \quad (1)$$

It is assumed that random variables ξ_i are statistically independent and have a normal distribution with zero expectation and a singular mean square deviation

$$w(x) = e^{-x^2/2} \Big/ \sqrt{2\pi} . \quad (2)$$

The number of terms N is the number of degrees of freedom.

In the second case

$$\gamma_2 = F_{N_1 N_2} = \frac{\chi_{N_1}^2}{N_1} \Bigg/ \frac{\chi_{N_2}^2}{N_2} /. \quad (3)$$

Random variables $\chi_{N_1}^2$ and $\chi_{N_2}^2$ are assumed to be statistically independent, and have chi-square distribution with the number of degrees of freedom N_1 and N_2 respectively.

Chi-square distribution and Fisher distribution are tabulated in the form of quantiles γ_p, determined from the condition

$P[\gamma \le \gamma_p]=p$. (4)

In [4] it is stated that for the results of observations the use of relations with distributions other than those for which these relations are obtained, can lead to significant errors. In this paper, based on statistical modeling it is shown that the difference between the real distribution of individual observations from the normal leads to a significant change in quantiles (4) for the statistics γ_1 and γ_2. In the simulation, arrays of random numbers were formed with the same distributions as in [5].

Objective. *The objective of the authors is to investigate distribution of statistics in hypotheses testing.*

Methods. *The authors use general scientific methods, comparative method, simulation, evaluation approach, specific statistical methods.*

Results.

Rectangular distribution is given by

$$w(x) = \begin{cases} 1/2b & |x| \le b; \\ 0 & |x| > b. \end{cases} \quad (5)$$

For this distribution $\sigma = b/\sqrt{3} \approx 0{,}58b$.

Triangular distribution is given by

$$w(x) = \begin{cases} (1-|x|/b)/b & |x| \le b; \\ 0 & |x| > b. \end{cases} \quad (6)$$

In this case $\sigma = b/\sqrt{6} \approx 0{,}41b$.

Trapezoidal, rectangular and triangular distributions in contrast to the normal (non-zero on the entire real axis) are finite, that is, they are different from zero in a finite interval. In the region of a maximum attained at x = 0, the normal distribution is flatter than triangular, and less flat than rectangular.

As a nonfinite distribution, less flat and decreasing with increasing module of the argument slower than normal, can be considered a bilateral exponential distribution (Laplace distribution)

$$w(x) = e^{-|x|/m} \big/ 2m . \quad (7)$$

For it $\sigma = m\sqrt{2} \approx 1{,}41m$.

As a nonfinite distribution, flatter and decreasing with increasing module of the argument faster than normal, we consider

$$w(x) = 2e^{-x^4/m^4} \big/ mG(0{,}25) \approx 0{,}552 e^{-x^4/m^4} \big/ m. \quad (8)$$

In this case $\sigma = m\sqrt{G(0{,}75)/G(0{,}25)} \approx 0{,}58\, m$, *where G(.) is complete gamma function. This distribution will be called rapidly decreasing in what follows.*

Random numbers with distributions (6), (7) and (8) were formed by the nonlinear transformation of numbers with uniform distribution generated within the program Excel. Type of transformation is given in [5].

In Table 1 for some N and p values quantiles of the chi-square distribution are given, corresponding to statistics γ_1 for a normal distribution of the results of individual observations.

In the simulation for each term in (1) 1000 values of numbers were formed with a given distribution, and according to processing results values of quantiles for the statistics γ_1 were obtained. The calculation results are shown in Table 2.

With uniform, triangular and rapidly decreasing distributions of single measurements quintiles are higher than in a normal distribution, with probabilities of 0,02-0,2 and less (almost everywhere) – with probabilities 0,8-0,98. The largest differences are observed in a uniform distribution, the smallest are observed in the triangular distribution. For Laplace distribution the situation is reversed. With the increasing number of degrees of freedom for all distributions there is a reduction of relative differences between confidence limits of the results for a normal distribution.

If the hypothesis of normal distribution of single measurement is performed, the statistics γ_2, defined by the formula (3) has Fisher distribution. Quantiles of this distribution for some p and numbers of degrees of freedom are shown in Table 3.

At default of assumptions of normality, as in the previous case, quantiles of the statistics γ_2 are not different. Simulation results for distributions (5) – (8) are given in Tables 4-7.

By comparing data in Tables 4-7 with Table 3 we see that for uniform, triangular and rapidly decreasing distributions of individual observations quantiles of the statistics γ_2 reduce as compared to quantiles of a normal dis-

Table 1

Tabular quantiles of chi-square distribution

p	The number of degrees of freedom N					
	2	3	5	10	20	30
0,02	0,040	0,185	0,752	3,059	9,237	16,306
0,05	0,103	0,352	1,145	3,94	10,851	18,493
0,1	0,211	0,584	1,61	4,865	12,444	20,599
0,2	0,446	1,005	2,343	6,179	14,578	23,364
0,8	3,219	4,642	7,289	13,442	25,038	36,25
0,9	4,605	6,251	9,236	15,987	28,412	40,256
0,95	5,991	7,815	11,070	18,307	31,410	43,773
0,98	7,824	9,837	13,388	21,161	35,020	47,962

Table 2

Quintiles of the statistics γ_1 for various distributions

P_{con}	The number of degrees of freedom N					
	2	3	5	10	20	30
Uniform distribution						
0,02	0,07	0,31	1,25	4,68	11,95	19,60
0,05	0,19	0,66	1,88	5,51	13,57	21,96
0,1	0,40	1,04	2,50	6,40	14,90	23,59
0,2	0,79	1,61	3,28	7,50	16,55	25,80
0,8	3,08	4,36	6,63	12,40	23,40	34,26
0,9	3,69	5,15	7,63	13,64	25,38	36,57
0,95	4,33	5,64	8,36	14,87	26,96	38,39
0,98	4,89	6,29	9,25	16,02	28,44	40,31
Triangular distribution						
0,02	0,04	0,16	0,78	3,21	9,93	17,51
0,05	0,10	0,36	1,16	4,21	11,59	19,62
0,1	0,22	0,58	1,84	5,20	13,14	21,59
0,2	0,45	1,09	2,55	6,50	15,17	24,25
0,8	3,43	4,77	7,13	13,13	24,57	35,71
0,9	4,57	5,94	8,53	15,27	27,35	39,02
0,95	5,39	6,81	9,85	17,00	29,87	41,74
0,98	6,28	8,00	11,24	19,36	32,19	44,49
Laplace distribution						
0,02	0,01	0,07	0,35	1,83	6,10	11,07
0,05	0,04	0,17	0,63	2,48	7,53	14,17
0,1	0,10	0,30	0,95	3,25	9,42	16,36
0,2	0,23	0,57	1,54	4,49	11,84	19,71
0,8	2,90	4,60	7,28	14,12	26,38	38,65
0,9	5,04	7,15	10,57	18,69	32,25	45,40
0,95	7,57	9,81	14,13	22,85	37,39	51,70
0,98	11,12	14,33	18,73	28,16	45,83	60,12
Rapidly decreasing distribution						
0,02	0,06	0,25	1,02	3,98	10,89	18,01
0,05	0,15	0,54	1,56	4,89	12,41	20,81
0,1	0,32	0,85	2,12	5,76	14,01	22,61
0,2	0,65	1,35	2,92	7,04	16,04	25,06
0,8	3,27	4,59	6,99	12,75	24,02	35,17
0,9	4,25	5,57	8,26	14,47	26,42	38,06
0,95	4,93	6,44	9,27	16,09	28,69	40,40
0,98	5,73	7,32	10,46	17,56	30,75	42,91

Table 3

Quintiles of Fisher distribution

N_2	p	N_1					
		2	3	5	10	20	30
2	0,75	2,10	2,07	2,03	2,05	2,04	2,05
	0,90	4,71	4,54	4,62	4,69	4,61	4,68
	0,95	9,47	9,51	8,65	10,13	9,42	9,56
	0,99	72,34	56,98	55,23	49,61	44,67	44,52
3	0,75	2,28	2,36	2,41	2,44	2,46	2,47
	0,90	5,46	5,39	5,31	5,23	5,18	5,17
	0,95	9,55	9,28	9,10	8,79	8,66	8,62
	0,99	30,8	29,5	28,2	27,2	26,7	26,5
5	0,75	1,85	1,88	1,89	1,89	1,88	1,88
	0,90	3,78	3,62	3,45	3,30	3,21	3,17
	0,95	5,79	5,41	5,05	4,74	4,56	4,50
	0,99	13,3	12,1	11,0	10,1	9,55	9,38
10	0,75	1,60	1,60	1,59	1,55	1,52	1,51
	0,90	2,92	2,73	2,52	2,32	2,20	2,16
	0,95	4,10	3,71	3,33	2,98	2,77	2,70
	0,99	7,56	6,55	5,64	4,85	4,41	4,25
20	0,75	1,49	1,48	1,45	1,40	1,36	1,34
	0,90	2,59	2,38	2,16	1,94	1,79	1,74
	0,95	3,49	3,10	2,71	2,35	2,12	2,04
	0,99	5,85	4,94	4,10	3,37	2,94	2,78
30	0,75	1,45	1,44	1,41	1,35	1,30	1,28
	0,90	2,49	2,28	2,05	1,82	1,67	1,61
	0,95	3,32	2,92	2,53	2,16	1,93	1,84
	0,99	5,39	4,51	3,70	2,98	2,55	2,39

Table 4

Quintiles of the statistics γ_2 for uniform distribution

N_2	p	N_1					
		2	3	5	10	20	30
2	0,75	2,10	2,07	2,03	2,05	2,04	2,05
	0,90	4,71	4,54	4,62	4,69	4,61	4,68
	0,95	9,47	9,51	8,65	10,13	9,42	9,56
	0,99	72,34	56,98	55,23	49,61	44,67	44,52
3	0,75	1,79	1,74	1,70	1,75	1,70	1,69
	0,90	3,45	3,20	3,11	3,15	3,16	3,07
	0,95	5,22	5,01	5,23	5,44	5,04	5,03
	0,99	13,69	12,35	11,31	13,74	13,00	12,54
5	0,75	1,65	1,55	1,48	1,47	1,46	1,43
	0,90	2,52	2,38	2,13	2,18	2,11	2,12
	0,95	3,35	3,24	3,03	2,85	2,73	2,75
	0,99	6,14	6,18	6,01	4,71	5,38	5,15
10	0,75	1,51	1,46	1,37	1,32	1,30	1,27
	0,90	2,18	1,96	1,76	1,73	1,64	1,63
	0,95	2,65	2,34	2,10	2,03	1,92	1,87
	0,99	3,96	3,65	3,05	2,71	2,71	2,58
20	0,75	1,48	1,40	1,29	1,26	1,19	1,19
	0,90	2,02	1,81	1,63	1,56	1,45	1,41
	0,95	2,31	2,11	1,87	1,76	1,58	1,54
	0,99	3,02	2,71	2,41	2,13	2,05	1,96
30	0,75	1,49	1,38	1,29	1,26	1,19	1,16
	0,90	1,94	1,77	1,60	1,52	1,39	1,35
	0,95	2,28	2,04	1,82	1,64	1,53	1,46
	0,99	2,82	2,57	2,28	1,92	1,77	1,70

Table 5

Quintiles of the statistics γ_2 for triangular distribution

N_2	p	N_1					
		2	3	5	10	20	30
2	0,75	2,62	2,81	2,93	3,07	3,13	3,15
	0,90	8,73	8,33	8,19	9,00	9,07	8,91
	0,95	20,15	21,30	18,92	16,45	17,04	17,02
	0,99	80,57	100,89	101,53	98,77	105,69	90,87
3	0,75	2,24	2,30	2,29	2,23	2,37	2,32
	0,90	5,13	5,30	4,71	4,77	4,84	4,83
	0,95	9,06	8,38	7,93	8,06	8,16	7,92
	0,99	31,30	34,16	26,95	20,96	19,82	20,45
5	0,75	1,84	1,83	1,78	1,76	1,76	1,74
	0,90	3,52	3,38	3,10	2,91	2,85	2,91
	0,95	5,80	5,05	4,21	3,97	3,94	3,82
	0,99	11,47	11,33	12,06	11,11	8,28	9,47
10	0,75	1,64	1,56	1,48	1,48	1,43	1,41
	0,90	2,86	2,52	2,24	2,15	1,98	1,89
	0,95	3,75	3,31	2,92	2,71	2,38	2,36
	0,99	5,57	4,48	4,14	4,19	3,71	3,70
20	0,75	1,55	1,47	1,40	1,35	1,30	1,28
	0,90	2,41	2,17	1,96	1,80	1,67	1,59
	0,95	3,06	2,78	2,41	2,15	1,92	1,86
	0,99	4,77	3,96	3,34	3,15	2,61	2,51
30	0,75	1,53	1,42	1,34	1,29	1,25	1,23
	0,90	2,32	2,06	1,90	1,77	1,59	1,51
	0,95	2,94	2,52	2,25	2,03	1,78	1,65
	0,99	4,19	3,77	3,05	2,84	2,30	2,17

Table 6

Quintiles of the statistics γ_2 for Laplace distribution

N_2	p	N_1					
		2	3	5	10	20	30
2	0,75	4,21	4,67	4,82	6,07	6,09	6,07
	0,90	15,62	17,70	16,80	19,22	19,39	19,16
	0,95	35,24	40,96	39,57	48,16	45,64	42,31
	0,99	361,01	332,19	346,67	274,46	230,41	252,57
3	0,75	2,94	3,16	3,46	4,07	4,25	4,18
	0,90	9,79	10,35	10,84	11,33	10,19	10,91
	0,95	19,31	20,91	20,52	20,56	21,18	20,97
	0,99	65,52	70,25	60,63	61,81	54,77	62,25
5	0,75	2,08	2,23	2,39	2,66	2,73	2,80
	0,90	5,85	5,49	5,13	5,89	5,77	5,59
	0,95	9,72	10,67	9,64	9,10	8,77	8,86
	0,99	30,87	26,60	21,68	21,96	23,87	25,57
10	0,75	1,61	1,67	1,72	1,92	1,90	1,88
	0,90	3,97	3,67	3,32	3,38	3,34	3,37
	0,95	6,41	5,83	5,15	5,12	4,60	4,42
	0,99	15,20	12,08	10,16	8,94	9,94	8,58
20	0,75	1,33	1,44	1,51	1,52	1,52	1,52
	0,90	3,34	2,90	2,63	2,58	2,33	2,34
	0,95	4,73	4,04	4,03	3,45	3,06	2,91
	0,99	9,07	7,84	5,85	5,68	4,93	4,31
30	0,75	1,28	1,43	1,48	1,49	1,43	1,44
	0,90	2,99	2,69	2,55	2,38	2,15	2,06
	0,95	4,34	4,26	3,58	3,10	2,74	2,52
	0,99	8,31	7,91	5,65	4,48	4,19	3,62

Table 7

Quintiles of the statistics γ_2 for rapidly decreasing distribution

N_2	p	N_1					
		2	3	5	10	20	30
2	0,75	2,46	2,49	2,39	2,48	2,47	2,48
	0,90	5,74	5,95	5,60	5,84	5,58	5,65
	0,95	11,52	10,90	10,66	11,98	11,21	11,70
	0,99	91,84	77,08	75,49	64,68	58,45	59,64
3	0,75	1,99	1,97	1,93	2,05	2,01	2,00
	0,90	4,21	4,04	3,92	3,86	3,84	3,81
	0,95	6,22	6,52	6,52	6,47	6,39	6,16
	0,99	18,93	17,03	15,36	16,73	15,08	14,66
5	0,75	1,76	1,69	1,61	1,61	1,64	1,63
	0,90	3,01	2,82	2,55	2,61	2,53	2,52
	0,95	4,21	4,06	3,62	3,67	3,38	3,32
	0,99	8,23	8,56	6,92	6,00	6,71	6,43
10	0,75	1,58	1,52	1,45	1,39	1,39	1,35
	0,90	2,50	2,26	1,97	1,98	1,84	1,85
	0,95	3,30	2,82	2,46	2,41	2,23	2,14
	0,99	5,16	4,58	3,88	3,36	3,31	3,21
20	0,75	1,51	1,45	1,36	1,32	1,24	1,23
	0,90	2,27	2,02	1,79	1,66	1,53	1,49
	0,95	2,73	2,36	2,19	1,98	1,78	1,70
	0,99	3,78	3,30	2,92	2,53	2,34	2,20
30	0,75	1,50	1,42	1,36	1,32	1,22	1,21
	0,90	2,24	1,96	1,74	1,62	1,48	1,43
	0,95	2,68	2,28	2,01	1,86	1,67	1,61
	0,99	3,45	3,23	2,75	2,21	1,99	1,90

tribution. The largest differences are observed in a uniform distribution, the smallest – in the triangular. In case of Laplace distribution quintiles increase.

As the number of degrees of freedom for all distributions increases, as in the previous case, there is a reduction of the relative differences from the results for a normal distribution.

Inaccuracy in values of quantiles of the statistics γ_1 and γ_2, given in Tables 2, 4—7, as shown in [5], is from a few hundredths to 0 or two tenths, which is quite acceptable.

Changes in quantiles of the statistics γ_1 and γ_2 due to differences of results' distribution from the normal leads to a change in the probability of errors of the first kind in hypothesis testing, if the decision rule is based on the assumption of normality of the results of observation. With an increase in quantiles these probabilities reduce, equivalent to a decrease in the level of thresholds when deciding on truth or falsity of the hypothesis being tested. Otherwise, they increase, which is equivalent to increasing the level of significance.

Conclusion. *The findings could be useful in assessing the reliability of the results of statistical hypotheses testing, unless there is confidence in compliance with the assumption of normality of distribution of individual observations.*

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БЮЛЛЕТЕНЬ ОСЖД: МОНГОЛЬСКИЙ СЕГМЕНТ

ОРГАНИЗАЦИЯ СОТРУДНИЧЕСТВА ЖЕЛЕЗНЫХ ДОРОГ (ОСЖД)

Очередной номер бюллетеня Организации сотрудничества железных дорог (2015, № 3) открывает блок материалов, посвященных железнодорожному транспорту Монголии – страны, стоявшей наряду с другими у истоков создания ОСЖД 60 лет назад.

В заглавной статье министр дорог и транспорта страны Т. Намхай знакомит читателей с нынешним состоянием монгольских железных дорог, их местом и ролью в транспортном комплексе государства. Автор подчеркивает, что на фоне глобализации международного рынка и развивающихся интеграционных процессов формируются все новые экономические и социальные потребности, а вместе с ними возникают и общими усилиями, с участием стран-партнеров, реализуются серьезные транспортные проекты.

Эту тему продолжает начальник Улан-Баторской железной дороги П. Лувсандагва, чье выступление подкрепляет подборка данных об истории создания первой в стране железнодорожной организации, строительстве пионерных линий, современном этапе деятельности, программах технической модернизации и международного сотрудничества АО «УБЖД».

Ретроспективный взгляд на тенденции становления железнодорожного транспорта в Монголии и отношения отраслевых структур с ОСЖД на протяжении 60 лет ее существования демонстрирует в журнальной статье Р. Раш, который занимал в свое время пост министра дорог и транспорта, был членом Великого хурала (парламента).

В рамках подготовки железнодорожного саммита и международной грузовой конференции ОСЖД, организатором которых в Сеуле выступила железнодорожная корпорация Республики Корея (KORAIL), в бюллетене рассказывается об открытии в 1899 году первой в стране железной дороги Гёнин-сон, соединявшей столицу и Инчхон, дальнейшем развитии отрасли, вплоть до сегодняшних высокоскоростных трасс и ближайших планах интеграции с евразийской системой железнодорожного сообщения.

Традиционные информационные разделы журнала касаются недавних событий и фактов из жизни железных дорог разных стран, включая организацию и проведение международных встреч, семинаров и выставок текущего года.

OSJD BULLETIN: MONGOLIAN SEGMENT

The new issue of the bulletin of Organization for Cooperation of Railways (2015, № 3) opens with a flow of materials devoted to rail transport of Mongolia – a country that stood, together with other, at the origins of OSJD 60 years ago.

In the cover story the Minister of Roads and Transport of the country T. Namhay acquaints readers with the current state of Mongolian railways, their place and role in the transport sector of the state. The author emphasizes that in the context of globalization of the international market and developing integration processes new economic and social needs generate, and by combined efforts, with the participation of partner countries major transport projects are implemented.

This topic is further supported by the head of Ulaanbaatar Railway P. Luvsandagva whose speech is reinforced with the collection of data about the history of the country's first railway company, construction of pioneer lines, present stage of activities, programs of technical modernization and international cooperation of JSC «Ulaanbaatar Railway».

A retrospective look at the tendencies of the railways in Mongolia and relations of industrial structures with OSJD during 60 years of its existence are demonstrated in the article of R. Rush, who held at the time a post of minister of roads and transport, was a member of the Great Khural (Parliament).

Within preparation for the railway summit and International Freight Conference of OSJD, organized in Seoul by Railway Corporation of the Republic of Korea (KORAIL), the bulletin describes the opening in 1899 of the country's first railway Gening-song, which connects the capital and Incheon, further development of the industry, up to today's high-speed lines and short-term plans of integration with the Eurasian railway system.

Traditional information sections of the journal refer to recent events and facts in the life of railways in different countries, including the organization of international meetings, seminars and exhibitions of this year.

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Бюллетень ОСЖД: монгольский сегмент
OSJD Bulletin: Mongolian Segment