MATHEMATICAL MODEL OF ASYNCHRONOUS MACHINE FOR VIBRATION RESEARCH

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ABSTRACT

The mathematical model, proposed by the authors, is based on a concept of asynchronous machine in a form of two infinitely long cylindrical shells with current layers, the shells being separated by an air circular gap. The study is based on comprehensive use of information relating to the operation mode, when axes of stator and rotor coincide. New parameters are introduced, resulting from radial displacement of a rotating rotor, but their formulation allows using known methods of calculating parameters of machines.

Keywords: mathematical model, vibration studies, induction machine, asynchronous machine, magnetic bearing, rotor, eccentricity, current layer, perturbed field, unperturbed field.

Background. *In vibration studies of asynchronous machines, which use magnetic suspension of a rotor, it is necessary to take into account the possibility of radial displacement of a rotating rotor. In this regard, its position in the mathematical model of the asynchronous or induction machine (hereinafter – IM) is defined by two coordinates respectively rotational one and radial one.*

Objective. *The objective of the authors is to develop a mathematical model of the induction machine to be applied in vibration studies.*

Methods. *The authors use general scientific and engineering methods, electrical engineering methods, mathematical methods (e.g. Laplace equation, Kirchhoff equation).*

Results. *The model is based on a representation of an IM in a form of infinitely long cylindrical shells with current layers (Pic. 1), the shells being separated by an air circular gap (circular with no radial displacement of the rotor or eccentric if any). Thus, on the cylindrical shell of radius ρ_1 there is a current layer simulating stator windings of the induction machine, with a density*

$j_1 = j_{m1} \sin(\omega_1 t - \theta)$,

where $j_{m_1} = \frac{3}{2} I_{m_1}$, I_{m_1} is the amplitude of the phase current of the stator, ω_1 is angular frequency of the stator current.

It is assumed that magnetic cores of stator and rotor (its radius is designated as ρ_2) have an infinite magnetic permeability and zero conductivity.

First, we find a magnetic field in the area of air circular gap $\rho_2 \le \rho \le \rho_1$. The calculation will be performed for the vector potential of the magnetic field, which in this area has a single component A, directed along a cylinder and satisfying in cylindrical coordinates the Laplace equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial \theta^2} = 0 \qquad (1)$$

for boundary conditions: $\rho = \rho_1 \to B_\theta = -\frac{\partial A}{\partial \rho} = \mu_0 j_1$,

$\rho = \rho_2 \to B_\theta = 0$, *that is*

$$\left.\begin{array}{l} \left.\dfrac{\partial A}{\partial \rho}\right|_{\rho=\rho_1} = \mu_0\left(-j_c \cos\theta + j_s \sin\theta\right) \\ \left.\dfrac{\partial A}{\partial \rho}\right|_{\rho=\rho_2} = 0, \end{array}\right\}, \qquad (2)$$

where $j_c = j_{m1} \sin \omega_1 t$, $j_s = j_{m1} \cos \omega_1 t$.

If we assume that $A = R(\rho) \times S(\theta)$, then the equation (1) decomposes into two equations:

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} - n^2 R = 0;$$

$$\frac{d^2 S}{d\theta^2} + n^2 S = 0,$$

where n^2 is a constant. The solutions of those equations are

$R = c\rho^n + d\rho^{-n}$, $S = g \cos(n\theta) + h \sin(n\theta)$,

then

$A = (c\rho^n + d\rho^{-n}) \times [g \cos(n\theta) + h \sin(n\theta)]$.

In this equation with boundary conditions (2) a constant n = 1, and values c, d, g and h, which do not depend on variables ρ and θ, satisfy the equations

$$c - d\frac{1}{\rho_2^2} = 0, \quad \left(c - d\frac{1}{\rho_1^2}\right) g = -\mu_0 j_c, \quad \left(c - d\frac{1}{\rho_1^2}\right) h = \mu_0 j_s.$$

Hence we get

$$cg = -\frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\mu_0 j_c, \qquad ch = \frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\mu_0 j_s,$$

$$dg = -\frac{(\rho_1 \rho_2)^2}{\rho_1^2 - \rho_2^2}\mu_0 j_c, \qquad dh = \frac{(\rho_1 \rho_2)^2}{\rho_1^2 - \rho_2^2}\mu_0 j_s$$

and for the vector potential we get an equation

$$A = \mu_0 \frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\left(\rho + \frac{\rho_2^2}{\rho}\right)\left(-j_c \cos\theta + j_s \sin\theta\right). \qquad (3)$$

Thus, components B_ρ and B_θ of magnetic induction vector (B = rot A) and scalar magnetic potential V (H = – grad V) will be equal to:

$$B_\rho = \frac{1}{\rho}\frac{\partial A}{\partial \theta} = \mu_0 \frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\left(1 + \frac{\rho_2^2}{\rho^2}\right)\left(j_c \sin\theta + j_s \cos\theta\right);$$

$$B_\theta = -\frac{\partial A}{\partial \rho} = -\mu_0 \frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\left(1 - \frac{\rho_2^2}{\rho^2}\right)\left(-j_c \cos\theta + j_s \sin\theta\right);$$

$$V = -\frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\left(\rho - \frac{\rho_2^2}{\rho}\right)\left(j_c \sin\theta + j_s \cos\theta\right) + V_0,$$

where V_0 is a constant. Or

$$B_\rho = \mu_0 j_{m1} \frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\left(1 + \frac{\rho_2^2}{\rho^2}\right)\cos\left(\omega_1 t - \theta\right); \qquad (4)$$

$$B_\theta = \mu_0 j_{m1} \frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\left(1 - \frac{\rho_2^2}{\rho^2}\right)\sin\left(\omega_1 t - \theta\right). \qquad (5)$$

Let's find a flux function U, using one of expressions

$$\frac{\partial U}{\partial \rho}=\frac{1}{\rho}\frac{\partial V}{\partial \theta}=-H_{\theta} \text{ or } \frac{1}{\rho}\frac{\partial U}{\partial \theta}=-\frac{\partial V}{\partial \rho}=H_{\rho}.$$

We get

$$U=\frac{\rho_1^2}{\rho_1^2-\rho_2^2}\times\left(\rho+\frac{\rho_2^2}{\rho}\right)\times\left(-j_c\cos\theta+j_s\sin\theta\right)+U_0,$$

U_0 is a constant. Now it is possible to calculate a complex potential $\xi_w=U+jV$ of a magnetic field in the area of circular air gap $\rho_2\leq\rho\leq\rho_1$ in the plane w:

$$\xi_w=\frac{\rho_1^2}{\rho_1^2-\rho_2^2}\left[\left(j_c+jj_s\right)w+\left(j_c-jj_s\right)\frac{\rho_2^2}{w}\right]+\xi_0, \qquad (6)$$

where $w=\rho e^{i\theta}$, ξ_0 is a constant, $j=\sqrt{-1}$.

Let's consider a magnetic field in the area of eccentric air gap in the plane z (Pic. 2). Here ε is eccentricity, points P_1 and P_2 with coordinates $(x_1, 0)$ and $(x_2, 0)$ are symmetric with respect to both circles (traces of cylindrical shells), i.e.

$$x_1x_2=\rho_1^2, \left(x_1-\varepsilon\right)\times\left(x_2-\varepsilon\right)=\rho_2^2.$$

Function, realizing the conformal mapping of the eccentric gap in the circular ring, is linear fractional function [1] and has a form

$$w=\lambda\times\frac{z-x_1}{z-x_2}, \qquad (7)$$

where λ is a constant, x_1 and x_2 are roots of the equation

$$\varepsilon x^2-\left(\rho_1^2-\rho_2^2+\varepsilon^2\right)x+\varepsilon\rho_1^2=0. \qquad (8)$$

Since $z=re^{j\phi}$ then $z-x_1=b_1e^{j\psi_1}$, $z-x_2=b_2e^{j\psi_2}$

where

$$b_1=\sqrt{r^2+x_1^2-2rx_1\cos\phi}, \quad \sin\psi_1=\frac{r\sin\phi}{b_1},$$
$$b_2=\sqrt{r^2+x_2^2-2rx_2\cos\phi}, \quad \sin\psi_2=\frac{r\sin\phi}{b_2}. \qquad (9)$$

Therefore, the mapping function w can be written as

$$w=be^{j\psi}, \qquad (10)$$

where $b=\lambda\frac{b_1}{b_2}$ and $\psi=\psi_1-\psi_2$, those quantities being functions of the coordinates r and ϕ of the current point in the plane z. Bringing (10) in (6), we get an expression for a complex potential of the magnetic field in the area of eccentric gap in the plane z:

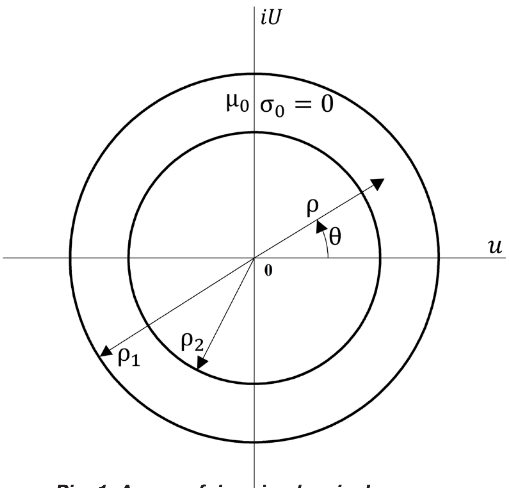
$$\xi_z=-\frac{\rho_1^2}{\rho_1^2-\rho_2^2}\left[\left(j_c+jj_s\right)be^{i\psi}+\left(j_c-jj_s\right)\frac{\rho_2^2}{be^{i\psi}}\right]+\xi_0. \qquad (11)$$

Hence we find the correlation for flux function and a component of the magnetic induction in this area

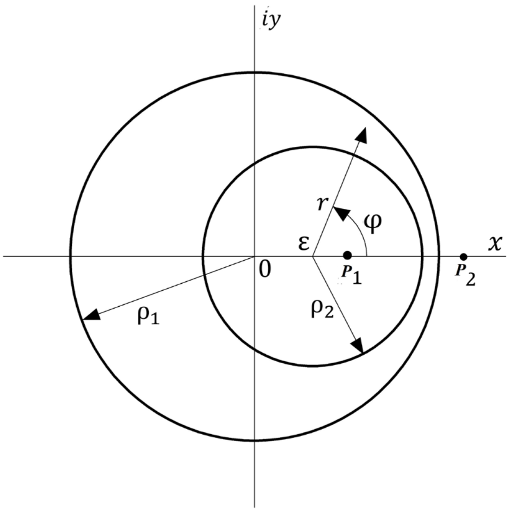
$$U=-\frac{\rho_1^2}{\rho_1^2-\rho_2^2}\left(b+\frac{\rho_2^2}{b}\right)\left(j_c\cos\psi-j_s\sin\psi\right); \qquad (12)$$

$$B_r=-\frac{\mu_0\rho_1^2}{\left(\rho_1^2-\rho_2^2\right)r}\left\{\begin{array}{l} j_c\frac{\partial}{\partial\phi}\left[\left(b+\frac{\rho_2^2}{b}\right)\cos\psi\right]- \\ -j_s\frac{\partial}{\partial\phi}\left[\left(b+\frac{\rho_2^2}{b}\right)\sin\psi\right]\end{array}\right\}; \qquad (13)$$

$$B_\phi=\frac{\mu_0\rho_1^2}{\rho_1^2-\rho_2^2}\left\{\begin{array}{l} j_c\frac{\partial}{\partial r}\left[\left(b+\frac{\rho_2^2}{b}\right)\cos\psi\right]- \\ -j_s\frac{\partial}{\partial r}\left[\left(b+\frac{\rho_2^2}{b}\right)\sin\psi\right]\end{array}\right\}. \qquad (14)$$



Pic. 1. A case of ring circular air clearance.



Pic. 2. A case of eccentric ring circular air clearance.

Let's consider the expressions (13) and (14) in a more detailed manner.

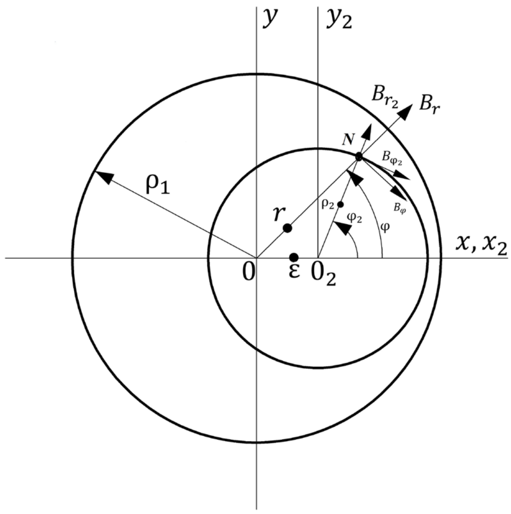
Let's note that the roots x_1 and x_2 of the equation (8), if the eccentricity ε is changing, vary in a very wide range. For example, when ε approaches its upper limit, i.e. when $\varepsilon\to(\rho_1-\rho_2)$, roots x_1 and x_2 are identical and equal to $x_1=x_2=½\times(\rho_1+\rho_2+\varepsilon)$; in contrast, when ε approaches its lower limit ($\varepsilon\to0$), one of the roots approaches zero, while the second root approaches infinity. Let's consider a case of small eccentricity, when $\varepsilon<<(\rho_1-\rho_2)$. Here, approximate values of roots of the equation (8) are given by expressions

$$x_1=\varepsilon\frac{\rho_1^2}{\rho_1^2-\rho_2^2}, \quad x_2=\frac{\rho_1^2-\rho_2^2}{\varepsilon}-\varepsilon\frac{\rho_2^2}{\rho_1^2-\rho_2^2}.$$

In this case $x_2>>r$, wherein $\rho_2<<r\leq\rho_1$; therefore we can neglect z in comparison with x_2 and write (7) in a form

$$w=\lambda\frac{z-\varepsilon'}{-x_2}, \quad \varepsilon'=\varepsilon\frac{\rho_1^2}{\rho_1^2-\rho_2^2}.$$

When $\varepsilon\to0$, obviously, $w\to z$ should take place; consequently, we can take $\lambda=-x_2$. Thus, for $\varepsilon<<(\rho_1-\rho_2)$



Pic. 3. Used coordinate systems.

the mapping function can be approximated as follows $w = z - \varepsilon'$. (15)

In this case values b and Ψ in (10)-(14) are equal to

$$b = r^2 - 2r\varepsilon' \cos\phi + \varepsilon'^2,$$
$$\sin\psi = \frac{r \sin\phi}{r - \varepsilon' \cos\phi}. \quad (16)$$

With account for (16) the expressions (13) and (14) can be written as follows:

$$B_r = \mu_0 j_{m_1} \frac{\rho_1^2}{\rho_1^2 - \rho_2^2} \begin{bmatrix} \left(1 + \frac{\rho_2^2}{r^2}\right)\cos(\omega_1 t - \phi) + \\ +2\varepsilon' \frac{\rho_2^2}{r^3}\cos(\omega_1 t - 2\phi) \end{bmatrix}, \quad (17)$$

$$B_\phi = \mu_0 j_{m_1} \frac{\rho_1^2}{\rho_1^2 - \rho_2^2} \begin{bmatrix} \left(1 - \frac{\rho_2^2}{r^2}\right)\sin(\omega_1 t - \phi) - \\ +2\varepsilon' \frac{\rho_2^2}{r^3}\sin(\omega_1 t - 2\phi) \end{bmatrix}. \quad (18)$$

If the magnetic field, described by the equations (4) and (5), is considered as unperturbed, then the field, corresponding in the expressions (17) and (18) to components, containing ε', should be considered as perturbed. In those expressions components without ε' coincide with (4) and (5) and represent the unperturbed field. Thus, the appearance of the eccentricity of the rotor is accompanied by a disturbance of the magnetic field in the air gap of the machine. For a small eccentricity of the rotor when $\varepsilon \ll (\rho_1 - \rho_2)$ perturbed magnetic field is rotating with a spatial period which is by half less than the period of the unperturbed field. This field rotates in the same direction as the unperturbed field at a speed equal to a half of the synchronous speed.

The equations (17) and (18) recorded in the reference frame xy, rigidly associated with a stator, with the start in the center of inertia of the stator (in the point 0 in Pic. 3). Let's calculate the magnetic field at the surface of the rotor (B_r and B_ϕ are components of magnetic induction in the point N at the surface of the rotor in the reference frame xy). Then let's write the results of the calculation (B_{r_2} and B_{ϕ_2}) in the coordinate system x_2y_2, stationary relative to the stator, but with the origin in the center of the rotor (in the point 0_2 in Pic. 3).

The result is

$$B_{r_2} = \mu_0 j_{m1} \begin{bmatrix} \frac{2\rho_1^2}{\rho_1^2 - \rho_2^2}\cos(\omega_1 t - \phi_2) + \\ \varepsilon_0 \mu_0 j_{m_1} \frac{2(\rho_1\rho_2)^2}{\rho_1^2 - \rho_2^2}\cos(\omega_1 t - 2\phi_2) \end{bmatrix}, \quad (19)$$

$$B_{\phi_2} = \varepsilon_0 \mu_0 j_{m_1} \frac{2\rho_1^2}{\rho_1^2 - \rho_2^2} \times \begin{bmatrix} \sin\omega_1 t - \\ \frac{\rho_1^2}{\rho_1^2 - \rho_2^2}\sin(\omega_1 t - 2\phi_2) \end{bmatrix}, \quad (20)$$

where $\varepsilon_0 = \frac{\varepsilon}{\rho_2}$ is relative eccentricity.

Let's assume that on the surface of the rotor there is a current layer, the density of which in the reference frame of the stator is given by the expression, which, according to our assumption, contains respectively unperturbed and perturbed components:

$$j_2 = j_{m_2} \sin(\omega_1 t - \phi_2 - \psi_2) + j_{m_\varepsilon} \sin(\omega_1 t - 2\phi_2 - \psi_\varepsilon). \quad (21)$$

The interaction of the magnetic field (19) and (20) (let's note that this field is due to the stator current) with the current sheet (21) gives the ponderomotive force acting on the rotor. The density of this force (per unit of surface of the rotor), taking into account the fact that $j_2 = -e_z j_2$, is:

$f = [j_2 B_2] = -e_\phi f_\phi + e_r f_r$;

$$f_\phi = j_2 B_{r_2}, \quad f_r = j_2 B_{\phi_2},$$

so that for the total force, with the active length of the machine l we have

$$F_\phi = l\rho_2 \int_0^{2\pi} f_\phi d\phi_2, \quad F_r = l\rho_2 \int_0^{2\pi} f_r d\phi_2.$$

Then tangential F_ϕ and rational F_r forces acting on the rotor, are:

$$F_\phi = \mu_0 j_{m_1} \frac{\rho_1^2}{\rho_1^2 - \rho_2^2} 2\pi l \rho_2 \begin{bmatrix} j_{m_2} \sin\psi_2 + \\ \varepsilon_0 j_{m_\varepsilon} \frac{\rho_2^2}{\rho_1^2 - \rho_2^2}\sin\psi_\varepsilon \end{bmatrix}, \quad (22)$$

$$F_r = \varepsilon_0 \mu_0 j_{m_1} j_{m_\varepsilon} \frac{\rho_1^4}{(\rho_1^2 - \rho_2^2)^2} 2\pi l \rho_2 \cos\psi_\varepsilon. \quad (23)$$

Values j_{m_1}, j_{m_2}, j_{m_ε}, ψ_2 and ψ_ε, which have been assumed till now to be given, are by and large unknown. To determine them, it is necessary to use the equations describing electromagnetic process in the machine. The possibility to directly use the known equations of the induction machine [2] for this purpose, obviously, needs to be justified. This is due to the fact that in this case, due to the eccentricity of the rotor, a disturbance of the magnetic field in the air gap of the machine occurs. Therefore, we need to figure out how this fact is reflected in the equations of the machine.

We first show that the equations for the values j_{m_1}, j_{m_2} and ψ_2 do not contain j_{m_ε} and ψ_ε, and those equations, in principle, coincide with the equations of the induction machine, which describe electromagnetic processes without the eccentricity of the rotor.

Let's write down (19) and (20) in the reference frame $x_f y_f$, rigidly connected to the rotor. To do this in (19) and (20) we make a substitution: $\phi_2 = \phi_f + \gamma$, where ϕ_f is an angular coordinate measured from the axis x_f,

$\gamma = \int_0^t \omega_2 dt + \gamma_0$, where ω_2 is a speed of rotor's rotation around its own axis. We have

$$B_{r_2} = \mu_0 j_{m_1} \frac{2\rho_1^2}{\rho_1^2 - \rho_2^2} \cos\left(\omega_1 t - \gamma - \phi_f\right) +$$

$$+\varepsilon_0 \mu_0 j_{m_1} 2\left(\frac{\rho_1 \rho_2}{\rho_1^2 - \rho_2^2}\right)^2 \cos\left(\omega t - 2\gamma - 2\phi_f\right). \qquad (24)$$

Let's calculate the flux linkage of the rotor coil winding, due to the field (24). The position of the coil is determined by the angle ϕ^0 between its magnetic axis and the axis x_f (Pic. 4):

$$\Phi\left(\phi^0\right) = \frac{\tau}{\pi} \int_{\phi^0 - \phi_{f_0}}^{\phi^0 + \phi_{f_0}} \int_{-l/2}^{l/2} B_{r_2} d\phi_f dz,$$

where τ is pole pitch, l is an active length of the machine. We obtain

$$\Phi\left(\phi^0\right) = \mu_0 j_{m_1} \frac{4\rho_1^2}{\rho_1^2 - \rho_2^2} \frac{\tau l}{\pi} \sin\phi_{f_0} \times$$

$$\times \left[\begin{array}{l} \cos\left(\omega_1 t - \gamma - \phi^0\right) + \\ \varepsilon_0 \dfrac{\rho_2^2}{\rho_1^2 - \rho_2^2} \cos\phi_{f_0} \cos\left(\omega_1 t - 2\gamma - 2\phi^0\right) \end{array} \right].$$

Then the flux linkage $\Psi(\phi^0)$ of the winding group is equal to

$$\Psi\left(\phi^0\right) = w_0 \int_{-\alpha_0/2}^{\alpha_0/2} \Phi\left(\phi^0 + \phi_f\right) d\phi_f.$$

Here $\Phi(\phi^0+\phi_f)$ is given as an expression for $\Phi(\phi^0)$ by replacing ϕ^0 in it to $(\phi^0+\phi_f)$, where ϕ^0 is now assumed as an angle, created by the axis x_f and the magnetic axis of the winding group, α_0 is an angle of phase zone ($\alpha_0 = \frac{\pi}{pm}$, p is a number of pole pairs, m is a number of phases), w_0 is a density of coils' distribution ($w_0 = \frac{q w_k}{\alpha_0}$, q is a number of slots per pole and phase, w_k is a number of coils in the winding). We have:

$$\Psi\left(\phi^0\right) = j_{m_1} \left[\begin{array}{l} m\cos\left(\omega_1 t - \gamma - \phi^0\right) + \\ m_\varepsilon \cos\left(\omega_1 t - 2\gamma - 2\phi^0\right) \end{array} \right], \qquad (25)$$

where

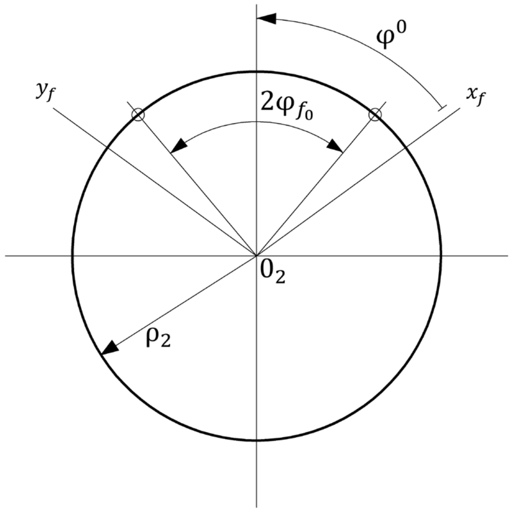
$$m = \mu_0 \frac{8\rho_1^2}{\rho_1^2 - \rho_2^2} \frac{\tau l}{\pi} w_0 \sin\phi_{f_0} \sin\frac{\alpha_0}{2}, \qquad (26)$$

$$m_\varepsilon = \varepsilon_0 m \frac{\rho_2^2}{\rho_1^2 - \rho_2^2} \cos\phi_{f_0} \cos\frac{\alpha_0}{2}. \qquad (27)$$

We write the equation of Kirchhoff for winding groups:

$$r_0 i_j + \frac{d\Psi_j}{dt} = 0; \; j = 1, 2, \ldots, 2n; n = mp. \qquad (28)$$

Flux linkage of the winding group Ψ_j is represented as the sum of linkages $\Psi_j(f)$ and $\Psi_j(s)$, resulting respectively from a field of the rotor and the stator field. Assigning the number 1 to a randomly chosen winding group, we enumerate all the others in the order of their location on the rotor's periphery. Let the coefficient of self-induction of the winding group be $l = l(0 \cdot \alpha)$, then the coefficient of mutual induction between j-th and k-th winding groups



Pic. 4. To the calculation of flux linkage of rotor winding coil.

with the same directions of the currents in them is:

$$l_{jk} = l\left(\left|j-k\right|\alpha\right) \text{ or } l_{jk} = l\left(\left(2n - \left|j-k\right|\right)\alpha\right),$$

where α is an angle between axes of adjacent winding groups ($\alpha = \frac{\pi}{n}$). At any phase currents in adjacent winding groups have opposite directions. Let in all odd winding groups currents are directed from the ear «n» to the ear «k», and in even windings from «k» to «n». Then for the flux linkage of the j-th coil group due to the rotor field, we get

$$\Psi_j(f) = \sum_{k=1}^{2n} l\left(\left|j-k\right|\alpha\right) i_k \left(-1\right)^{j+k}, \; j = 1, 2, \ldots, 2n.$$

The flux linkage $\Psi_j(s)$ is determined by the formula (30) at a value ϕ^0, corresponding to the j-th winding group. Let for the first group $\phi^0 = 0$, then for the j-th winding group we have

$\phi^0 = (j-1) \times \alpha\rho$ (29)

EMF corresponding to the flux linkage, calculated by the formula (25) with ϕ^0 according to (29), are directed in all winding groups equally from «n» to «k». The currents in odd winding groups have the same directions, while currents in even groups, as mentioned previously, have opposite direction. To match the direction of EMF and currents in even winding groups, it is necessary to attribute an opposite sign to the flux linkages calculated by (25) at (29) for even winding groups. Thus, we obtain the following expression for the flux linkage $\Psi_j(s)$, true for both even and odd winding groups:

$$\Psi_j(s) = j_{m_1}\left[m\cos\left(\omega_1 t - \gamma - \left(j-1\right)\alpha p\right) + \right.$$

$$\left. + m_\varepsilon \cos\left(\omega_1 t - 2\gamma - 2\left(j-1\right)\alpha p\right)\right]\left(-1\right)^{j-1}. \qquad (30)$$

Let the winding group number 1 enters a phase of «a». Then belongings of other groups are found at once. In the phase «a» we have 1, 4, ..., 2n-2 winding groups, in the phase «b» we have 3, 6, ..., 2n groups and in the phase «c» we have 2, 5, ..., 2n-1 groups.

Let's sum equations (28) for the values of the index i, which it takes in individual phases (assuming that winding groups in the phases are connected in series), and take into account:

$i_a = i_1 = i_4 = \cdots = i_{2n-2}$, $i_b = i_3 = i_6 = \cdots = i_{2n}$, $i_c = i_2 = i_5 = \cdots = i_{2n-1}$.

Then we get

$$\mathrm{Ri}+\frac{d}{dt}\left(\mathrm{Li}+\Psi(s)\right)=0, \qquad (31)$$

where i, $\Psi(s)$ are column matrices of currents and flux linkages of phases, justified by the stator magnetic field:

$$i=\left(i_a i_b i_c\right)^{\mathrm{T}},\ \ \Psi(s)=\left(\Psi_a(s)\Psi_b(s)\Psi_c(s)\right)^{\mathrm{T}},$$

R is a matrix of phase resistance, L is a matrix of phase inductances:

$$\mathrm{R}=\begin{pmatrix} r & 0 & 0\\ 0 & r & 0\\ 0 & 0 & r\end{pmatrix},\ \mathrm{L}=\begin{pmatrix} l_0 & l_m & l_m\\ l_m & l_0 & l_m\\ l_m & l_m & l_0\end{pmatrix}.$$

Here r is an active resistance of the phase, l_0 is self-inductance of the phase, l_m is mutual inductance between phases:

$$r=2pr_0,\ \ l_0=2p\sum_{k=0}^{2p-1}(-1)^k l(3k\alpha),$$

$$l_m=2p\sum_{k=0}^{2p-1}(-1)^k l\left((2+3k)\alpha\right).$$

Flux linkage of phases, justified by stator magnetic field, is calculated, referring to (30):

$$\Psi_a(s)=2pj_{m_1}m\cos(\omega_1 t-\gamma);$$

$$\Psi_b(s)=2pj_{m_1}m\cos(\omega_1 t-\gamma-120);$$

$$\Psi_c(s)=2pj_{m_1}m\cos(\omega_1 t-\gamma+120).$$

Hence, perturbed magnetic field, inducted by the rotor eccentricity, which in (30) corresponds to a component of flux linkage of the winding group, containing a comprising factor m_ε, does not contribute to the flux linkage of phases of the rotor winding. This is due to the fact that the period of the perturbed magnetic field distribution is by half less than the pole pitch. Therefore, in each phase at normal rotor winding flux linkage of odd winding groups due to perturbed magnetic field is compensated by the flux linkage of even winding groups of the same magnetic field. Thus, equations for variables j_{m_1}, j_{m_2} and Ψ_2 (they can be called equations of the unperturbed state) should not differ from the known equations of an induction machine, corresponding to the case of coincidence of the axes of stator and rotor.

Let's now define the values j_{m_ε} and Ψ_ε, associated with the perturbed magnetic field. In induction machines, with a rotor winding of a squirrel cage (either single or double), and in machines with a throttling-type or massive rotor, perturbed magnetic field, the radial component of which (see (24)) is

$$B_{r_\varepsilon}=2\varepsilon_0\mu_0 j_{m_1}\left(\frac{\rho_1\rho_2}{\rho_1^2-\rho_2^2}\right)^2\cos\left(\omega_1 t-2\gamma-2\phi_f\right), \qquad (32)$$

induces a current in the rotor. This current is associated with previously introduced value $j_{m_\varepsilon}\sin\left(\omega_1 t-2\phi_2-\psi_\varepsilon\right)$ (see (21)). In asynchronous machines with the specified rotor design the period of distribution of the current, induced in the rotor, coincides with the period of the external magnetic field B_{r_ε}. Therefore, the flux linkage of the «coil», corresponding to (36) and calculated by the formula

$$\Phi_\varepsilon=\frac{\tau}{\pi}\int_{\phi^0-45}^{\phi^0+45}\int_{-l/2}^{l/2}B_{r_\varepsilon}d\phi_f dz,$$

will be equal to:

$$\Phi_\varepsilon=2\varepsilon_0\mu_0 j_{m_1}\left(\frac{\rho_1\rho_2}{\rho_1^2-\rho_2^2}\right)^2\frac{\tau l}{\pi}\cos\left(\omega_1 t-2\gamma-2\phi^0\right).$$

If the rotor winding is made as a squirrel cage with the number of rods h, the flux linkage of the «phase» of the rotor can be determined by the formula

$$\Psi_\varepsilon=\frac{hp}{\pi}\int_{-\alpha_0/2}^{\alpha_0/2}\Phi_\varepsilon d\phi^0,$$

where p is a number of pole pairs, α_0 is an angle of phase zone.

We get

$$\Psi_\varepsilon=2\varepsilon_0\mu_0 j_{m_1}\left(\frac{\rho_1\rho_2}{\rho_1^2-\rho_2^2}\right)^2\frac{\tau lph}{\pi^2}\sin\alpha_0\cos\left(\omega_1 t-2\gamma\right)$$

As well as a «phase» EMF of the rotor e_ε, corresponding to Ψ_ε, which at a constant speed of rotor's rotation is equal to:

$$e_\varepsilon=2\varepsilon_0\mu_0 j_{m_1}\left(\frac{\rho_1\rho_2}{\rho_1^2-\rho_2^2}\right)^2\frac{\tau lph}{\pi^2}\omega_1(1-2s)\bullet$$
$$\sin\alpha_0\sin\omega_1(1-2s)t,$$

where s is motor slip ($s=\frac{\omega_1-\omega_2}{\omega_1}$). Hence we find the current

$$i_\varepsilon=I'_{m_\varepsilon}\sin\left[\omega_1(1-2s)t-\psi_\varepsilon\right], \qquad (33)$$

$$I'_{m_\varepsilon}=\frac{2\varepsilon_0\mu_0 j_{m_1}\left(\frac{\rho_1\rho_2}{\rho_1^2-\rho_2^2}\right)^2\frac{\tau lph}{\pi^2}\omega_1(1-2s)\sin\alpha_0}{\sqrt{R_\varepsilon^2+(1-2s)^2X_\varepsilon^2}}, \qquad (34)$$

$$\psi_\varepsilon=\mathrm{arctg}\frac{(1-2s)X_\varepsilon}{R_\varepsilon}, \qquad (35)$$

where R_ε is an active resistance, X_ε is an inductive resistance of leakage of rotor. By the way, for the experimental determination of those parameters it would have been necessary to perform a stator winding with a half less pole pitch with unchanged other conditions.

The current i_ε forms a symmetrical positive series system, where (33) can be regarded as an expression of one of the phase currents in the reference frame of the rotor. It is very important that the period of distribution of «winding» with the current i_ε is by half less than the pole pitch. Therefore, current sheet, equivalent to this «winding», in the reference frame of the stator is given by the formula

$$j_\varepsilon=j_{m_\varepsilon}\sin\left(\omega_1 t-2\phi_2-\psi_\varepsilon\right), \qquad (36)$$

which was set a priori in (21). In the derivation of the expression (36) it was assumed that the phase «windings» are distributed harmonically and are symmetrical relative to each other. In (36) $j_{m_\varepsilon}=\frac{3}{2}I'_{m_\varepsilon}$.

Let's find an expression for j_{m_ε}. Let's first note that in the case of squirrel-cage rotor (26) we should take $w_0=\frac{h}{2\pi}$, $\sin\phi_{f_0}=1$. If we further consider that $2pm = L_{12}$ is a coefficient of mutual inductance between the stator and the rotor, then, bearing in mind (31) and (38), we obtain

$$j_{m_\varepsilon} = \frac{\varepsilon_0 \dfrac{3\rho_2^2}{4\left(\rho_1^2-\rho_2^2\right)} \cos\dfrac{\alpha_0}{2}\,\omega_1\left(1-2s\right)}{\sqrt{R_\varepsilon^2+\left(1-2s\right)^2 X_\varepsilon^2}} L_{12} j_{m_1}. \qquad (37)$$

The equations (35) and (37) play a very important role. They define values and Ψ_ε, which are included in the formula for the radial component of the ponderomotive force acting on the rotor (23). Thereby the equation of radial movement of the rotor is concretized, and takes the form

$$M\frac{d^2\varepsilon}{dt^2} = 2\varepsilon\mu_0 j_{m_1} j_{m_\varepsilon} \frac{\rho_1^2}{\left(\rho_1^2-\rho_2^2\right)^2}\pi l\cos\psi_\varepsilon - F_c, \qquad (38)$$

where F_c is radial force of resistance, for example, of a magnetic bearing, M is a mass of the rotor.

To determine the values j_{m_1}, j_{m_2} and Ψ_2 it is necessary to solve equations of an induction machine, describing processes in the absence of the eccentricity of the rotor. Those equations in general form can be written as follows:

$$\left.\begin{aligned} &R_1 \mathrm{i}_1 + L_1\frac{d\mathrm{i}_1}{dt} + L_{12}\frac{d\mathrm{i}_2}{dt} = \mathrm{u}_1;\\ &R_2 \mathrm{i}_2 + L_2\frac{d\mathrm{i}_2}{dt} + L_{21}\frac{d\mathrm{i}_1}{dt} - j\omega L_2 \mathrm{i}_2 - j\omega L_{21}\mathrm{i}_1 = 0;\\ &J\frac{d\omega}{dt} = M_\phi - M_c. \end{aligned}\right\} \qquad (39)$$

Here u_1, i_1, i_2 are mapping vectors of voltage and current; R_1 and R_2 are active resistances of stator and rotor; $L_{12} = L_{21}$ is their mutual inductance, equal to the main inductance of stator winding; $L_1 = L_{12} + L_{\sigma_1}$, $L_2 = L_{21} + L_{\sigma_2}$, L_{σ_1} and L_{σ_2} are leakage inductances of stator and rotor, including slot, end-coil and differential leakages; M_ϕ is a moment, justified by tangential component of the ponderomotive force; M_c is a moment of resistance; J is a moment of inertia relative to the axis of rotation of the rotor.

In harmonic distribution of phase windings and their symmetrical arrangement relative to each other the values j_{m_1} and j_{m_2} are numerically equal to the amplitudes I_{m_1} and I_{m_2} of currents i_1 and i_2, multiplied by $\frac{3}{2}$, and Ψ_2 is an angle between vectors i_1 and i_2. In the sinusoidal mode, assuming that in (39) $u_1 = \dot{U}_{m_1} e^{j\omega_1 t}$, $i_1 = \dot{I}_{m_1} e^{j\omega_1 t}$ and $i_2 = \dot{I}_{m_2} e^{j\omega_1 t}$, we can immediately calculate $\dot{I}_{m_1}$ and $\dot{I}_{m_2}$ by the equations:

$$\left.\begin{aligned} &\left(R_1 + jX_1\right)\dot{I}_{m_1} + jX_{12}\dot{I}_{m_2} = \dot{U}_{m_1}\\ &\left(R_2 + jX_2 s\right)\dot{I}_{m_2} + jX_{12}s\dot{I}_{m_1} = 0 \end{aligned}\right\}, \qquad (40)$$

which result from (39) at ω = const, and then we could calculate, using them, the values j_{m_1}, j_{m_2} and Ψ_2.

In small asynchronous machines, electromagnetic time constants are small, so if those machines are fed from the infinite buses (U_{m_1}, ω_1 = const) their dynamic properties do not differ from static features. In that case, the considered procedure to determine the values j_{m_1}, j_{m_2}, j_{m_ε}, Ψ_2 and Ψ_ε, based on the use of (37) and (40), is completely justified. In contrast, for large asynchronous machines with sound windings and circuits a possibility of significant transients should be borne in mind. This leads to the need instead of expressions (40) to use the equations (39), in which $M_\phi = j\times\frac{X_{12}}{2}\times(i_2\times\overset{*}{i}_1 - i_1\times\overset{*}{i}_2)$, and instead of formulas (34) and (35) to use equations

$$L_\varepsilon\frac{di_\varepsilon}{dt} + R_\varepsilon i_\varepsilon = e_\varepsilon;$$

$$e_\varepsilon = -2\varepsilon_0\mu_0\left(\frac{\rho_1\rho_2}{\rho_1^2-\rho_2^2}\right)^2\frac{\tau lph}{\pi^2}\sin\alpha_0\frac{d}{dt}\left[j_{m_1}\cos\left(\omega_1 t - 2\gamma\right)\right], \qquad (41)$$

where j_{m_1} is already a time function.

Conclusions. *Thus, to solve the problem of investigating the induction machine with magnetic bearings, and in particular, to solve the problem of calculating its vibration characteristics, an asynchronous machine can be simulated by the equations (38) and (39), (41); or equations (35), (37), (38) and (40).*

Finally, let's note that the parameters in the equation (39) and (40) can be easily calculated by known methods [3] or found experimentally [4]. With regard to the parameters R_ε and X_ε, they are new and, as already noted, represent active and inductive leakage of the rotor, provided that the number of poles of the machine doubled. The calculation of those parameters according to methods used in the practice of the engineering and designing of electrical machines, apparently, is not difficult. However, the experimental determination of those parameters in the existing machines is unlikely to be feasible.

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