USE OF FOURIER INTEGRAL TRANSFORM IN CALCULATION OF STRUCTURES

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ABSTRACT

This paper presents a method to obtain a system of linear equations describing a final element in the frame structure. In solution process integral Fourier transform and generalized functions are used. The boundary conditions at the ends of the finite element are determined by the theorem of Paley–Wiener–Schwartz. Application of the method for calculation of beam, cable-stayed bridges and extradosed bridges allows to set all kinds of loads without replacing them with any equivalent.

Keywords: bridge, superstructure, reinforced concrete structure, finite element method, generalized functions, differential equation of Timoshenko beam, integral Fourier transform, theorem of Paley–Wiener–Schwartz.

Background. *The vast majority of calculations of beam, cable-stayed and extradosed bridges are performed using the finite element method (FEM). This method is based on decomposition of a calculated design into a number of small but finite in size elements – finite element (FE) and replacement of differential equations describing its stress-strain state with a system of linear algebraic equations.*

Despite its broad capacities, FEM has a number of disadvantages [1, p. 39]:

1) For each of a final element it is necessary to constitute a matrix of rigidity.

2) Distribution of the load must be replaced by equivalent generalized nodal forces.

3) Concentrated forces should be applied at the nodes.

4) It is necessary to carry out additional calculations to obtain the values of a bending moment and shear forces on the borders of the finite element.

5) It is necessary to divide the design into small finite elements in order to ensure calculation accuracy.

If a calculated design can be replaced by a frame system, these disadvantages can be eliminated. To do this, it is necessary to get a system of linear equations describing the finite element, using integral Fourier transform, finite generalized functions [4, 6] and theorem of Paley–Wiener–Schwartz. Details of the method are presented in the thesis of E. N. Kurbatsky [5].

In his work, E. N. Kurbatsky used the method in construction of finite element for a beam on an elastic foundation without taking into account longitudinal force and transverse shear.

The article describes an example of finding the model finite element of reinforced concrete beam of bridge superstructure:

– Taking into account longitudinal and shear forces impact;

– Without taking into account longitudinal forces.

Objective. *The objective of the author is to consider use of Fourier integral transform in calculation of structures.*

Methods. *The author uses general scientific and engineering methods, mathematical apparatus, graph construction, analysis.*

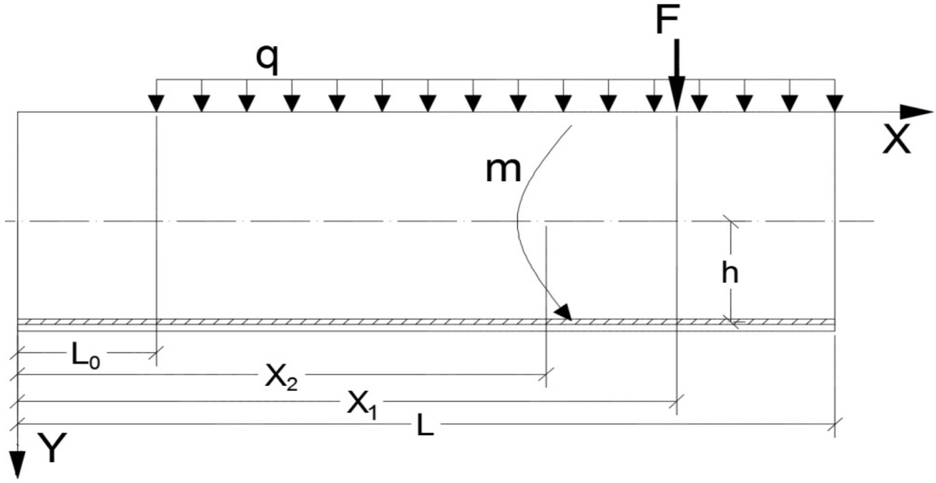
Results.

I.

We obtain a system of equations describing the finite element shown in Pic. 1. To do this, we use the differential equation of Timoshenko beam, which takes into account the effect of longitudinal and shear forces on the deflection [5]:

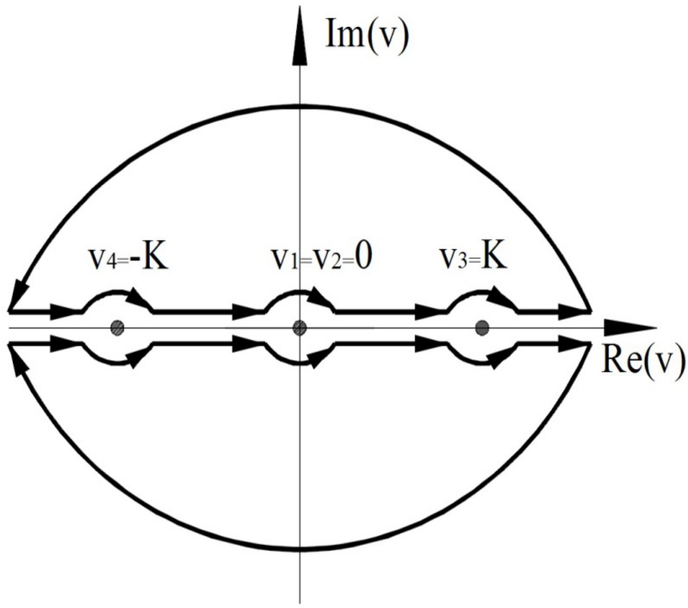
$$EI\frac{d^4u}{dx^4}+P_i\frac{d^2u}{dx^2}=q(x)+\frac{k}{GA}EI\frac{d^3Q_x}{dx^3}, \qquad (1)$$

where G is deformation module;



Pic. 1.

Pic. 2.



A is area of cross-section;

k is dimensionless coefficient that depends in the shape of cross-section [7]. For a rectangular cross-section:

$$k = \frac{12 + 11\upsilon}{10(1+\upsilon)} . \qquad (2)$$

The equation (1) should be represented in finite generalized functions. To do this, we write the deflection function and the function of lateral force:

U(x) = u(x)[H(x)-H(x-l)]; (3)

$$\overline{Q}(x) = Q(x)[H(x) - H(x-l)] , \qquad (4)$$

where H(x) is Heaviside function.

Then the equation (1) gets the following form [2, 3]:

$$\begin{aligned}
& EI\frac{d^4U}{dx^4} + P_i\frac{d^2U}{dx^2} = EI\frac{d^4u}{dx^4}[H(x) - \\
& -H(x-l)] + P_i\frac{d^2u}{dx^2}[H(x) - H(x-l)] - \\
& -EIn\frac{d^3Q_x}{dx^3}[H(x) - H(x-l)] + Q(x)\delta(x) - \\
& -Q(x)\delta(x-l) + M(x)\delta'(x) - M(x)\delta'(x-l) + \\
& +EI\phi(x)\delta''(x) - EI\phi(x)\delta''(x-l) + \\
& +EIu(x)\delta'''(x) - EIu(x)\delta'''(x-l) + \\
& +P(\phi(x)\delta(x) - \phi(x)\delta(x-l) + \\
& +u(x)\delta'(x) - u(x)\delta'(x-l)) - \qquad (5) \\
& -EIn(Q(x)\delta''(x) - Q(x)\delta''(x-l) + \\
& +Q'(x)\delta'(x) - Q'(x)\delta'(x-l)),
\end{aligned}$$

where $n = \frac{k}{GA}$. (6)

In the equation (5) the feature of delta-function is used:

$$\int_{-\infty}^{\infty} f(x)\delta(x-t)dx = f(t) . \qquad (7)$$

The impact of prestressing reinforcement on the beam can be represented in the form of bending moments m, applied at the breakage of armature and longitudinal compressive force P.

Let's denote the right side of the equation (5), which is a generalized load S(x), and $\frac{P_i}{EI} = K_i^2$ *, then:*

$$\frac{d^4U}{dx^4} + K_i^2\frac{d^2U}{dx^2} = \frac{S(x)}{EI} . \qquad (8)$$

Let's apply Fourier transform to both sides of the equation:

$$\int_{-\infty}^{\infty}\left[\frac{d^4U}{dx^4} + K^2\frac{d^2U}{dx^2}\right]e^{ivx}dx = \int_{-\infty}^{\infty}\frac{S(x)}{EI}e^{ivx}dx . \qquad (9)$$

We get [2]:

$$\tilde{U}(v)[v^4 - K^2v^2] = \frac{\tilde{S}(v)}{EI} , \qquad (10)$$

where $\tilde{U}(v)$ *is Fourier image of function U(x);*

$\tilde{S}(v)$ *is Fourier image of generalized load S(x);*

v is parameter of Fourier transform.

For preparation of the system of equations of the finite element we considered it is necessary to find Fourier image of generalized load [2, 3]:

$$\begin{aligned}
& \tilde{S}(v) = q(\frac{e^{ivl} - e^{ivl_0}}{iv}) + Fe^{ivx_1} + m(-iv)e^{ivx_2} + \\
& +Q(0) - Q(l)e^{ivl} + M(0)(-iv) - M(l)(-iv)e^{ivl} + \\
& +EI\phi(0)(-iv)^2 - EI\phi(l)(-iv)^2e^{ivl} + \\
& +EIu(0)(-iv)^3 - EIu(l)(-iv)^3e^{ivl} + \\
& +P_i(\phi(0) - \phi(l)e^{ivl} + \\
& +u(0)(-iv) - u(l)(-iv)e^{ivl}) - nEI(Q(l)v^2e^{ivl} - \qquad (11) \\
& -Q(0)v^2 + qiv(e^{ivl} - e^{ivl_0}) + \frac{F}{a}iv(e^{iv(x_1+a)} - e^{ivl(x_1-a)})).
\end{aligned}$$

Hence this method takes into account the impact of the shift, the concentrated forces should be represented as distributed load acting in the vicinity of the application points of respective concentrated forces. The length of the sections is taken equal to *half the height of the beam cross-section at the site. This must be done only in the term of equation (11), which is responsible for the account of shift under the action of concentrated forces, namely:*

$$\frac{F}{a} iv(e^{iv(x_1+a)} - e^{ivl(x_1-a)}) . \qquad (12)$$

From the equation (10) we get:

$$\tilde{U}(v) = \frac{1}{EI}\frac{\tilde{S}(v)}{v^4 - K^2 v^2} . \qquad (13)$$

Theorem of Paley–Wiener–Schwartz for generalized functions allows to find the unknown, because, according to it, $\tilde{U}(v)$ function must be integral and, consequently, the numerator is the sum of integral functions that will contain the zeros of the denominator [4, 6].

The following conditions must be met:

$\tilde{S}(v_j) = 0, j = 1,3,4;$ (14)

$\tilde{S}'(v_j) = 0, j = 2,$ (15)

where v_j is roots of the expression

$v^4 - K^2 v^2 = 0.$ (16)

Let's find these roots (Pic. 2):

$v_{1,2} = 0,\ v_3 = K,\ v_4 = -K.$ (17)

Substituting these roots in the equation (11), we obtain a system consisting of four equations and describing one final element shown in Pic. 1:

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & C_{1,5} & C_{1,6} & C_{1,7} & C_{1,8} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & C_{2,5} & C_{2,6} & C_{2,7} & C_{2,8} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} & C_{3,5} & C_{3,6} & C_{3,7} & C_{3,8} \\ C_{4,1} & C_{4,2} & C_{4,3} & C_{4,4} & C_{4,5} & C_{4,6} & C_{4,7} & C_{4,8} \end{bmatrix} \times \begin{bmatrix} Q(0) \\ Q(l) \\ M(0) \\ M(l) \\ \phi(0) \\ \phi(l) \\ u(0) \\ u(l) \end{bmatrix} = -\begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} \\ G_{2,1} & G_{2,2} & G_{2,3} \\ G_{3,1} & G_{3,2} & G_{3,3} \\ G_{4,1} & G_{4,2} & G_{4,3} \end{bmatrix} \begin{bmatrix} q \\ F \\ m \end{bmatrix}; \qquad (18)$$

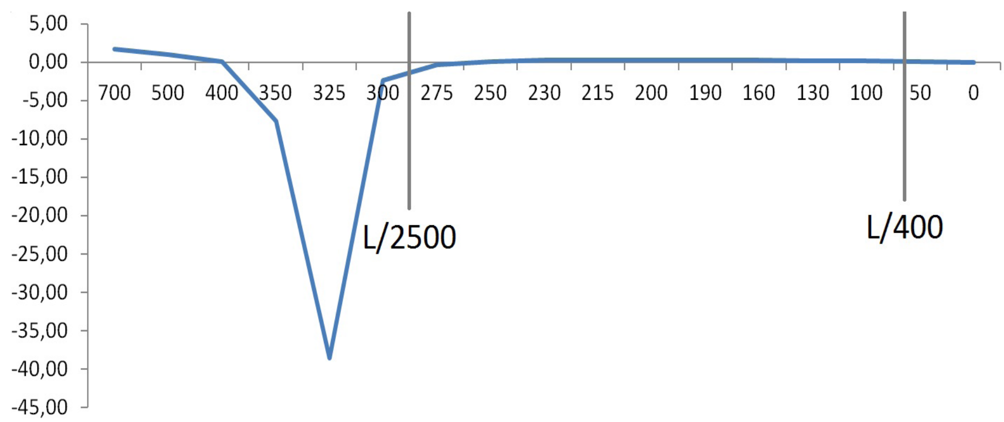
$$C_{1,1} = 1 + nEIv_j^2$$
$$C_{2,1} = 2nEIv_j$$
$$C_{1,2} = -e^{ivl} - nEIv_j^2 e^{ivl}$$
$$C_{2,2} = -ile^{ivl} - 2nEIv_j e^{ivl} - nEIv_j^2 ile^{ivl}$$
$$C_{1,3} = -iv$$
$$C_{2,3} = -i$$
$$C_{1,4} = ive^{ivl}$$
$$C_{2,4} = ie^{ivl} - lve^{ivl}$$
$$C_{1,5} = -EIv^2 + P$$
$$C_{2,5} = -2EIv$$
$$C_{1,6} = EIv^2 e^{ivl} - Pe^{ivl}$$
$$C_{2,6} = 2EIve^{ivl} + EIv^2 ile^{ivl} - Pile^{ivl}$$
$$C_{1,7} = EIiv^3 - Piv$$
$$C_{1,7} = 3EIiv^2 - Pi$$
$$C_{1,8} = -EIiv^3 e^{ivl} + Pive^{ivl}$$
$$C_{1,8} = -3EIiv^2 e^{ivl} + EIlv^3 e^{ivl} + Pie^{ivl} - Plve^{ivl}$$
$$G_{1,1} = \left(\frac{e^{ivl} - e^{ivl_0}}{iv}\right) + nEIiv(e^{ivl} - e^{ivl_0})$$
$$G_{2,1} = \mathrm{i}\frac{e^{ivl} - e^{ivl_0}}{v^2} - \mathrm{i}\frac{ile^{ivl} - il_0 e^{ivl_0}}{v} + \qquad (19)$$
$$+ nEI(ie^{ivl} - lve^{ivl} - ie^{ivl_0} + l_0 ve^{ivl_0})$$
$$G_{1,2} = e^{ivx_1}$$
$$G_{2,2} = ix_1 e^{ivx_1}$$
$$G_{1,3} = -ive^{ivx_2}$$
$$G_{2,3} = -ie^{ivx_2} + vx_2 e^{ivx_2} .$$

The difference between the first, third and fourth equations is only the multiplier v_j.

Since roots $v_{1,2}$ are always zero, it is necessary instead of the multiplier at q to write (20) for the first equation and (21) – for the second:

$$\lim_{v\to 0}(\mathrm{q}\frac{e^{ivl} - e^{ivl_0}}{iv}) = \mathrm{q}(\mathrm{l} - \mathrm{l}_0) ; \qquad (20)$$

$$\lim_{v\to 0}(\mathrm{qi}\frac{e^{ivl} - e^{ivl_0}}{v^2} - \mathrm{qi}\frac{ile^{ivl} - il_0 e^{ivl_0}}{v}) = \mathrm{qi}(\frac{\mathrm{l}^2}{2} - \frac{\mathrm{l}_0^2}{2}) . \qquad (21)$$



Pic. 3.

Thus, the system of equations describing a finite element has the form:

$$CU = -GN. \qquad (22)$$

Let's compare the results of the calculation of the beam obtained using Midas Civil software system and the method. As an example, we consider a prestressing reinforced concrete beam 50 m long of rectangular cross-section 2x0,3 m on two supports, loaded with distributed load with intensity q = 1t / m and a concentrated force F = 10 t, applied at distance x1 = 0,25L from the edge of the beam.

In Pic. 3 vertical axis corresponds to a discrepancy in the calculation, expressed as a percentage, and the horizontal axis – longitudinal force in the prestressing reinforcement, expressed in tonnes. If the deflection of the beam tends to zero, the difference reaches significant values. In the graph it can be seen at the site where the longitudinal force is 325 tons. However, it is necessary to take into account that in all other cases, when the deflection is u = L / 2500 ÷ L / 400, the difference is not more than-half per cent that allows to obtain exact solutions in the calculation of real structures. In addition, it should be noted that the difference in bending moment and shear force at any value of the longitudinal force gives an error of less than one percent.

II.

When there is no need to consider the longitudinal force, the system of equations takes a much simpler form.

The differential equation of the deflection of the beam:

$$EI\frac{d^4u}{dx^4} = q(x) + \frac{k}{GA}EI\frac{d^3Q_x}{dx^3}. \qquad (23)$$

After Fourier transform we get:

$$\tilde{U}(v)[v^4] = \frac{\tilde{S}(v)}{EI}; \qquad (24)$$

$$\tilde{U}(v) = \frac{1}{EI}\frac{\tilde{S}(v)}{v^4}; \qquad (25)$$

$$v^4 = 0. \qquad (26)$$

Fourier image of generalized load:

$$\begin{aligned}\tilde{S}(v) &= q(\frac{e^{ivl}-1}{iv}) + Fe^{ivx_1} + m(-iv)e^{ivx_2} + Q(0) - \\ &-Q(l)e^{ivl} + M(0)(-iv) - M(l)(-iv)e^{ivl} + \\ &+EI\phi(0)(-iv)^2 - EI\phi(l)(-iv)^2e^{ivl} + \\ &+EIu(0)(-iv)^3 - EIu(l)(-iv)^3e^{ivl} - \\ &-nEI(Q(l)v^2e^{ivl} - Q(0)v^2 + qiv(e^{ivl}-1)).\end{aligned} \qquad (27)$$

Let's find roots of the equation:

$$v_{1,2,3,4} = 0. \qquad (28)$$

The following equations should be met:

$$\tilde{S}(v_1) = 0, \quad \dot{\tilde{S}}(v_2) = 0, \quad \ddot{\tilde{S}}(v_3) = 0, \quad \dddot{\tilde{S}}(v_4) = 0. \qquad (29)$$

Matrices C and G get the following form:

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -il & -i & i & 0 & 0 & 0 & 0 \\ 2nEI & l^2 - 2nEI & 0 & -2l & -2EI & 2EI & 0 & 0 \\ 0 & il^3 - 6nEIil & 0 & -3il^2 & 0 & 6EIil & 6EIi & -6EIi \end{bmatrix}; \qquad (30)$$

$$G = \begin{bmatrix} (l - l_0) & 1 & 0 \\ i(\frac{l^2}{2} - \frac{l_0^2}{2}) & ix_1 & -i \\ (\frac{-l^3}{3} + \frac{l_0^3}{3}) - 2nEI(l - l_0) & -x_1^2 & 2x_2 \\ i(\frac{-l^4}{4} + \frac{l_0^4}{4}) - 3inEI(l^2 - l_0^2) & -ix_1^3 & 3ix_2^2 \end{bmatrix}. \qquad (31)$$

This model is useful primarily because there is no need each time to substitute in the matrices C and G values $v_{1,2,3,4} = 0$, as in the system of equations (18). The advantage of the model is the fact that the matrix C is tenuous.

Conclusion.

1. The application of the systems of equations when calculating beam, cable-stayed bridges and extradosed bridges allows to set any type of load without a need to replace them with equivalent loads.

2. The applied method does not require additional partition of structure into finite elements in order to obtain an exact solution when the moment of inertia is constant.

3. Matrix C is the equivalent of stiffness matrix in finite element method is unchanged for all finite elements in a case where there is no longitudinal load. If its account is required, in the matrix should be substituted the values of obtained roots of the equation

$$v_{1,2} = 0, v_3 = \sqrt{\frac{P_i}{EI}}, v_4 = -\sqrt{\frac{P_i}{EI}}.$$

4. In the method of constructing a system of linear equations, focused on a finite element in the frame system, for unknown are taken displacement, rotation angle of the cross section, bending moment and shear force at the ends of the finite element. This allows to immediately find the exact value of these variables without the need for additional calculations.

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