ON ESTIMATION OF RAILWAY TRACK STATE

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ABSTRACT

Building hardware complex diagnostics of railway track on the basis of analysis of dynamic processes in the movement of train does not have alternatives at the moment. Similar methods have long been used in the operation of aircraft and other fields of technology and mechanical engineering, biomedical field. The ability to simplify calculations of model tasks is provided by authors' additions to the theory of basic models of «Timoshenko beam» in different situations. Inhomogeneous system of linear differential equations with partial derivatives of the second order is accurately reduced to sequence of solution of two classic mixed problems – derived equations: hyperbolic and Klein-Gordon-Fock. It is shown that the system has two scales (two basic frequencies). The effect of abnormally rapid fluctuations oscillations is explained. A semi-analytical – numerical method is offered that allows to compare effects caused by different boundary conditions.

Keywords: diagnostics of railway track, principles of system construction, Timoshenko beam models, partial differential equations, abnormally rapid fluctuations oscillation, effects comparison.

Background. *In connection with development of high-speed rail transportation has become of particular importance creation and operation of high-tech complex to control quality of track state directly from the moving track. The theoretical basis for this method of control is set out in [1–5] and scientific works cited in them. It is important not to contrapose different ways of diagnostics, they should not compete, but complement each other. Methods of integrating them into a single transport system are intended to be a guide for experts, simulating the processes of defect detection [6].*

Objective. *The objective of the authors is to estimate state of railway track using mathematical methods.*

Methods. *The authors use general scientific and engineering methods, comparison, mathematical methods.*

Results.

1. Preliminary notes

Our main goal is to identify previously unknown mathematical properties of main theoretical models of beams (sleepers), which play an important role in the calculations associated with analysis of vibration spectrum when rolling stock is moving on tracks to present ideas on the principles of building control systems.

Remark 1. *We note that the experience of computer processing and analysis of large volumes of dynamic data with complex three-dimensional internal structure is stored, for example, in the use of different types of brain imaging, processing of neuroimaging and bioinformatics data in the biomedical field. Mathematical statistics methods and comparison with standards are used, which leads to success. There are a lot of similarities with our problem, although in it waves and vibrations in different frequency ranges are studied. In both cases there is a priori information about the structure of objects.*

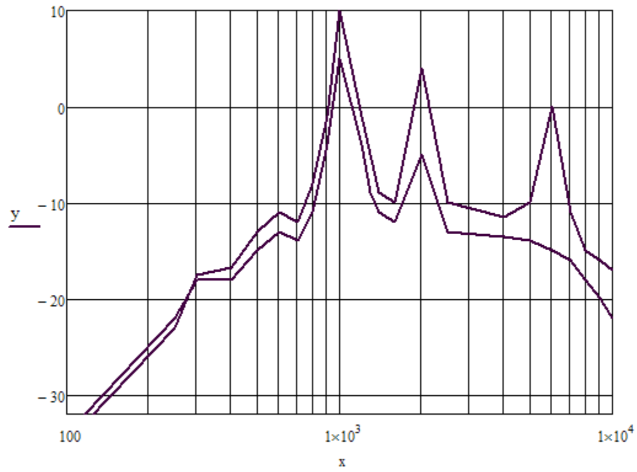
Note 2. *The famous mathematician, expert in the field of dynamical systems A. S. Bratus expressed his support for establishment of a complex, which monitors the state of the track from rolling stock, on the basis of currently available technical equipment, high-tech, computing systems, GLONASS system. Mathematical calculations of fluid flow in complex branched pipes, a wide range of temperatures and pressures, he said, still remains a challenge. Attraction for numerical calculation supercomputers and multithreaded data processing methods requires a lot of forces, energy and financial resources. But this does not mean that we should be afraid to look for solutions on a collision course. It is not without self-irony scientist says that if people were waiting for mathematicians to prove all their theorems related to emerging challenges, and the water pipeline might not exist today.*

The task to identify defects that cause changes in the spectrum of longitudinal, vertical and transverse vibrations occurring during the passage of rolling stock on rails, is called «reverse» in mathematics. Multi-dimensional «direct» boundary value problems with a variety of diverse options for setting boundary conditions and different distributions in the space of forces are usually very complex. «Reverse» tasks of such complexity cannot be solved currently. From our point of view, we can restrict the calculation of one-dimensional, simpler «direct» tasks. Mathematical modeling in this case, allows to identify main effects and to have tips that can be used to conclude on appearance of hard-detectable defects and to locate them in space. Early detection of defects helps to avoid accidents, loss of property, not to mention the loss of health and life of humans.

In [1–5], there are numerical studies of dynamics of interaction of the train with sleepers, which are considered as beams. They show that the presence of voids under sleepers and roadbed destruction significantly affect the distribution of dynamic loads and consequently on oscillation spectrum.

Many calculations, both ours and of other authors, suggest that the emergence of defect leads to additional resonance and dispersion in the vicinity of main peaks increases. Similar repetitive features may become a basis of a method to identify hard-detectable recognizable defects, they can localize track section by comparing the recorded spectrum of transverse, vertical and longitudinal vibrations with respective reference spectra obtained previously on the serviceable section.

As an example one of simple oscillation spectra and dependence of acceleration from frequency and presence of various defects is shown in Pic. 1. Summing up the results of such variants of numerical mathematical modeling, we can conclude that an important effect has been found: with increase in the number of sleepers, «hanging» on a rail, that is deprived of support, the dispersion in the vicinity of frequencies of major resonance increases and additional resonances appear at higher frequencies.



Pic. 1. Acceleration of rail (db / g) due to oscillation frequency (Hz).

The lower curve in Pic. 1 reflects an ideal situation, when there are no defects. The upper curve corresponds to the presence of two «suspended» sleepers, without relying on ballast. Similar curves of spectra for other defects are in other mentioned works.

Note 3. *In numerical modeling specialists use implicit difference schemes and the Ritz variation method for boundary value problems. In analytical construction of solutions by decomposition into Fourier series [7], the authors lament the fact that for the embodiment with sufficient accuracy it is necessary to summarize 32000 terms. That is, the Fourier series converges slowly.*

Note 4. *Numerical experiments for solution of direct multi-dimensional problems have been described in a number of mentioned works. A wide variety of properties, geological, geographical, related to infrastructure of surrounding area and as a result a variety of possible boundary conditions makes dynamic mathematical «direct» models ineffective. The solution of «reverse» problems of such a complexity is a more difficult problem.*

Note 5. *A new element from the point of view of mathematical physics and mechanics of beams is factorization of partial differential equations of «Timoshenko beams» model. If a reader is aware of works, where such factorization is implemented, when please let the authors know. The younger generation, not knowing analytic operator methods immediately rushes to a computer.*

Factorization procedure makes it possible to combine a large number of tasks that are different in parameter values (because of differences in physical constants) and a number of terms in them, which value and influence on a solution can be adjusted using dimensionless parameters and boundary conditions. Formulas are obtained compact, easily foreseeable (six integral terms) and allowing to compare the effect of different boundary conditions. Obtaining numerical results is reduced to calculation of integrals using classic formulas.

Operator methods are described in the works of V. P. Maslov, M. V. Karasev, V. G. Danilov, A. Hadamard and others [8–12], where it is proved that factorization leads to consistency in solving classical problems for inhomogeneous linear hyperbolic equations and inhomogeneous linear Klein-Gordon-Fock equation, and show that there are two basic scales, two characteristic speeds, two base frequencies.

2. Statement of a problem

Equations describing oscillations in «Timoshenko beam» models have a view of a system [2, p. 15–38]:

$$\begin{gathered} E\,I\frac{\partial^2 \vartheta(x,t)}{\partial x^2} + k\,AG\left[\frac{\partial Z(x,t)}{\partial x} - \vartheta(x,t)\right] - \\ -m r^2 \frac{\partial^2 \vartheta(x,t)}{\partial t^2} + P\,\vartheta(x,t) = 0, \\ \frac{\partial}{\partial x}\left\{k\,AG\left[\frac{\partial Z(x,t)}{\partial x} - \vartheta(x,t)\right]\right\} - \\ -m\frac{\partial^2 Z(x,t)}{\partial t^2} = f(x,t). \end{gathered} \tag{1}$$

Here Z(x, t), Θ(x, t) are functions of deviation of midline and angle of rotation of the cross section, respectively;

E is elasticity modulus on compression [N/m²];

I is reduced moment of inertia of cross-section [m⁴];

k is dimensionless Timoshenko coefficient;

A is area of cross section [m²];

G is elasticity modulus on shift [N/m²];

$m = I\rho/r^2$ is distributed density;

ρ is specific density [kg/m];

P is longitudinal force applied to a beam.

Note that between dimensions of functions Z(x, t), Θ(x, t) the relation [Θ(x, t)] = [Z(x, t)] / [m] is correct. For the system (1) in the cited works are considered mixed boundary value problems in different situations.

3. Splitting of the system (1)

Let's show that mixed problems for the system (1) have an exact solution.

Theorem 1. *Let there be given a mathematical «Timoshenko beam» model (1). Then, one of the functions is clearly expressed through the other:*

$$\begin{gathered} Z(x,t) = \left[\begin{array}{l} -h(t) + \left[k\,AG - P\right]\Phi(x,t) + \\ + m r^2 \dfrac{\partial^2 \Phi(x,t)}{\partial t^2} - E\,I\dfrac{\partial^2 \Phi(x,t)}{\partial x^2} \end{array}\right] / (k\,AG), \\ \vartheta(x,t) = \frac{\partial \Phi(x,\,t)}{\partial x}. \end{gathered} \tag{2}$$

Here h(t) is arbitrary continuously differentiable function.

To determine the function Φ(x, t) it is necessary to solve a problem with boundary conditions and initial conditions for inhomogeneous linear fourth-order partial differential equation:

$$m[kAG-P]\frac{\partial^2\Phi(x,t)}{\partial t^2}+kAGP\frac{\partial^2\Phi(x,t)}{\partial x^2}+$$
$$+m^2r^2\frac{\partial^4\Phi(x,t)}{\partial t^4}-m(EI+kAGr^2)\frac{\partial^4\Phi(x,t)}{\partial x^2\partial t^2}+$$
$$+kAGEI\frac{\partial^4\Phi(x,t)}{\partial x^4}=F_0(x,t). \qquad (3)$$

Here $F_0(x,t)=-kAG\,(mh''(t)+f(x,t))$.

Proof.

Boundary conditions for the system (1) can be first, second or third kind. Since the first equation of the system (1) is non-uniform, and the function f(x, t) contains the sum of generalized Dirac delta functions in fixed and mobile points $x = v_0(t)t$*, which moves unevenly, the procedure for establishment of the fourth order equation is different from the establishment in case of a homogeneous second equation (1). The second equation cannot be differentiated. Let's differentiate the first equation of the system (1) with respect to x and express the second variable* $\frac{\partial^2 Z(x,t)}{\partial x^2}$*. We make a change*

$\vartheta(x,t)=\frac{\partial\Phi(x,t)}{\partial x}$ *and get*

$$kAG\frac{\partial^2Z(x,t)}{\partial x^2}+[P-kAG]\frac{\partial^2\Phi(x,t)}{\partial x^2}-$$
$$-mr^2\frac{\partial^4\Phi(\chi,t)}{\partial x^2\partial t^2}+EI\frac{\partial^4\Phi(x,t)}{\partial x^4}=0. \qquad (4)$$

Integrating (4) twice on variable x*, and we obtain the formula (2).*

The function h(t) occurs in integration. One of the constants of integration is set equal to zero. Thus, one function in the system (1) is expressed by the other – formula (2). After differentiating (2) with respect to t from the second equation (1) follows (3).

4. Application of factorization to the equation (3).

The aim of factorization procedure is to present the equation (3) in the form

$$L_1\circ\left[L_2\,Y(x,t)\right]=F_0(x,t).$$

Let's conduct nondimensionalization of the equation (3). Let's denote via $x = x_0\chi$, $t = t_0\tau$ *and* ψ *characteristic size, time and characteristic value of the function Φ(x, t) respectively. We make a change* $\Phi(x,t)=\Theta(\frac{x}{x_0},\frac{t}{t_0})/\psi$. *We denote hyperbolic D'Alembert operator with μ parameter that determines base speed of waves and base frequency:*

$$\Pi_\mu=\frac{\partial^2}{\partial\tau^2}-\mu^2\frac{\partial^2}{\partial\chi^2}. \qquad (5)$$

From further analysis it appears that there are two dimensionless parameters

$$a^2=\frac{Et_0^2}{x_0^2\rho},\ \mu^2=\frac{AGkr^2t_0^2}{Ix_0^2\rho}. \qquad (6)$$

Theorem 2. *Let's there is an equation (3) and parameters (6) are defined. Then, for construction of solutions of the equation it is necessary to solve sequentially two standard mixed problems, namely: for linear hyperbolic partial differential equation*

$$\Pi_a\,Y(\chi,\tau)=F_1(\chi,\tau),$$
$$F_1(\chi,\tau)=M_0(a^2-\mu^2)\,\psi\,[t_0^2\,EI\,h''(t_0\tau)+$$
$$+(arx_0)^2\,f(x_0\chi,t_0\tau)]/(EI\mu^2) \qquad (7)$$

and mixed problem for Klein-Gordon-Fock equation

$$\Pi_\mu\,\Theta(\chi,\tau)-x_0^2\,\mu^4\,\Theta(\chi,\tau)/(r^2(a^2-\mu^2))=$$
$$=-x_0^2\,\mu^4\,Y(\chi,\tau)/(M_0r^2(a^2-\mu^2)), \qquad (8)$$

Here $M_0 = const \neq 0$.

In the second embodiment factorization method gives result, which is demonstrated further.

Theorem 3. *Let there is an equation (3) and two dimensionless parameters (6). Then, for construction of solutions of the equation it is necessary to solve sequentially two standard mixed problems, namely: for Klein-Gordon-Fock equation*

$$\Pi_\mu\,Y(\chi,\tau)-x_0^2\,\mu^4\,Y(\chi,\tau)/(r^2(a^2-\mu^2))=F_2(\chi,\tau),$$
$$F_2(\chi,\tau)=x_0^2\,\mu^4\,c_0\,\psi\,[EIt_0^2\,h''(t_0\tau)+$$
$$+(arx_0)^2\,f(x_0\chi,t_0\tau)]/(a^2\,r^4(\mu^2-a^2)) \qquad (9)$$

and mixed problem for linear hyperbolic equation

$\Pi_a\Theta(x,\tau) = a^2r^2(a^2-\mu^2)\,Y(x,\tau)/(\mu^2c_0EI)$. *(10)*

Here $c_0 = const \neq 0$.

Proof of theorems 2, 3. *Let's consider Klein-Gordon-Fock equation for the function* $\Theta(x, \tau)$ *with undetermined constant coefficients* M_0, c_0P, j_0. *We find the second derivative with respect to* τ:

$$\frac{\partial^2\Theta(\chi,\tau)}{\partial\tau^2}=[c_0P\frac{\partial^2\Theta(\chi,\tau)}{\partial\chi^2}-M_0\Theta(\chi,\tau)+Y(\chi,\tau)]/j_0.$$

Thus $Y(x,\tau)$ *is a function that is determined by the problem for the equation (7) or (9).*

After calculating four derivatives of the function $\Phi(x,t)=\Theta(x/x_0,\,t/t_0)/\psi$ *we get*

$$\frac{\partial^4\Phi(x,t)}{\partial t^4}=\frac{\partial^4\Theta(\chi,\tau)}{\partial\tau^4}\frac{1}{t_0^4\psi},$$
$$\frac{\partial^4\Phi(x,t)}{\partial x^4}=\frac{\partial^4\Theta(\chi,\tau)}{\partial\chi^4}\frac{1}{x_0^4\psi},$$
$$\frac{\partial^4\Phi(x,t)}{\partial x^2\partial t^2}=\frac{\partial^4\Theta(\chi,\tau)}{\partial\chi^2\partial\tau^2}\frac{1}{x_0^2t_0^2\psi}.$$

Substituting in the equation (3) – and in the first case (for theorem 2) we have

$$j_0=\frac{mc_0Px_0^2}{kAGt_0^2},\quad c_0=\frac{M_0(kAGr^2-EI)}{(kAGPx_0^2)},$$
$$P=\frac{kAGEI}{(EI-kAGr^2)},\quad m=\frac{I\rho}{r^2}. \qquad (11)$$

After introduction of parameters (6) follow equations (7), (8).

In the second case (for theorem 3), we obtain

$$j_0=\frac{mc_0Pr^2x_0^2}{EIt_0^2},\ M_0=0,$$
$$P=\frac{kAG\,EI}{(EI-kAGr^2)},\ m=\frac{I\rho}{r^2}. \qquad (12)$$

After introduction of parameters (6) follow equations (9), (10).

Then it is necessary to set boundary and initial conditions and perform calculations of convolution integrals with the right side and boundary conditions by analytical and numerical methods. Calculations at constant values, relating to railway track [2, 3], show that the ratio of parameters (speed of waves and frequencies) $\mu / a > 50$. *Coefficient* $a^2-\mu^2<0$.

In $f(x_0\chi, t_0\tau)$ there are Dirac delta functions, so integral with them are easily calculated. Characteristic is the case described in theorem 2, when in the first phase of mixed problem (7) is solved with smaller parameter a. Then the calculation of the first task makes it possible to obtain the envelope of low-frequency vibrations. Calculation of convolution integrals of the second task (8) «fills» the resulting structure with high-frequency oscillations and little changes of the envelope curve of low-frequency vibrations. Thus, the analogy with amplitude modulated structure having two base frequencies, two scales is explained [9].

In [11, p. 260] there are formulas of the third boundary value problem for a linear hyperbolic equation with a speed a in case of a rod with elastic fixed endpoints with different stiffness coefficients k_1, k_2 and initial conditions $0 \le \chi \le l$.

Let's set the following initial conditions:

$$Y(\chi,0)=f_0(\chi),\quad \frac{\partial Y(\chi,\tau)}{\partial\tau}\Big|_{\tau=0}=f_1(\chi) \qquad (13)$$

and boundary conditions

$$\begin{aligned}&\frac{\partial Y(\chi,\tau)}{\partial\chi}\Big|_{\chi=0}-k_1\,Y(0,\tau)=g_1(\tau),\\&\frac{\partial Y(\chi,\tau)}{\partial\chi}\Big|_{\chi=l}-k_2\,Y(l,\tau)=g_2(\tau).\end{aligned} \qquad (14)$$

The solution $Y(\chi, \tau)$ is determined by the formula

$$\begin{aligned}Y(\chi,\tau)=&\frac{\partial}{\partial\tau}\int_0^l f_0(\xi)\,G(\chi,\xi,\tau)\,d\xi+\\&+\int_0^l f_1(\xi)\,G(\chi,\xi,\tau)\,d\xi+\int_0^\tau\int_0^l F_1(\xi,\tau)G(\chi,\xi,\tau-\theta)\,d\xi\,d\theta-\\&-a^2\int_0^\tau g_1(\tau)G(\chi,0,\tau-\theta)\,d\theta+a^2\int_0^\tau g_2(\tau)G(\chi,l,\tau-\theta)\,d\theta,\end{aligned} \qquad (15)$$

and Green function –according to the formula

$$G(\chi,\xi,\tau)=\frac{1}{a}\sum_{n=1}^{\infty}\left(\begin{aligned}&\frac{1}{(\lambda_n\|u_n\|^2)}\sin(\lambda_n\,\chi+\phi_n)\times\\&\times\sin(\lambda_n\,\xi+\phi_n)\sin(\lambda_n\,a\,\tau)\end{aligned}\right),$$

$$\phi_n=\arctan\left(\frac{\lambda_n}{k_1}\right), \qquad (16)$$

$$\|u_n\|^2=\frac{l}{2}+\left[\frac{(\lambda_n^2+k_1k_2)(k_1+k_2)}{2(\lambda_n^2+k_1^2)(\lambda_n^2+k_2^2)}\right],$$

where λ_m are positive roots of the transcendental equation $ctg(\lambda l) = (\lambda^2-k_1k_2)/(\lambda(k_1+k_2))$. The principal term of the expression for approximate positive eigenvalue has a form

$$\lambda=\frac{\sqrt{k_1+k_2+k_1k_2\,l}}{\sqrt{l}}. \qquad (17)$$

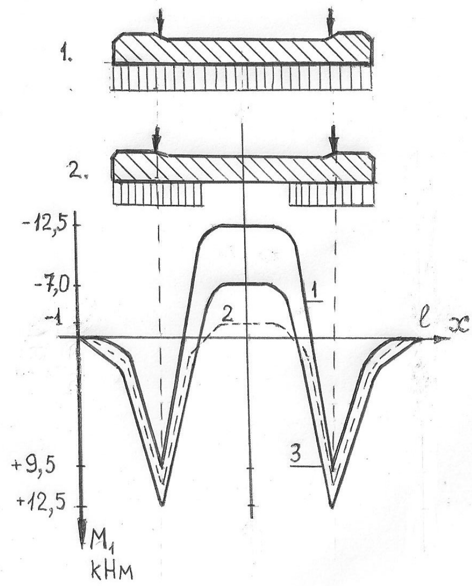
On the other hand in case of oscillations of the rod, one end of which is rigidly fixed and the other is free, we have a mixed boundary value problem. In [11, p. 261], there are formulas for a linear hyperbolic equation with a speed a in the range $0 \le \chi \le l$. Suppose that initial conditions (13) and boundary conditions

$$\frac{\partial Y(\chi,\tau)}{\partial\chi}\Big|_{\chi=0}=g_1(\tau),\quad \frac{\partial Y(\chi,\tau)}{\partial\chi}\Big|_{\chi=l}=g_2(\tau). \qquad (18)$$

The solution $Y(\chi, \tau)$ is determined by the formula, similar to (15), where the fourth term

$$+a^2\int_0^\tau g_1(\tau)\left[\frac{\partial}{\partial\xi}G(x,\xi,\tau-\theta)\right]\Big|_{\xi=0}d\theta.$$

Green function in this case has a form



Pic. 2. Response options of a bending moment of a sleeper, lying on the ballast.

$$G(\chi,\xi,\tau)=$$
$$=\frac{2}{l\,a}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sin(\lambda_n\,\chi)\sin(\lambda_n\,\xi)\sin(\lambda_n\,a\,\tau)\right). \qquad (19)$$

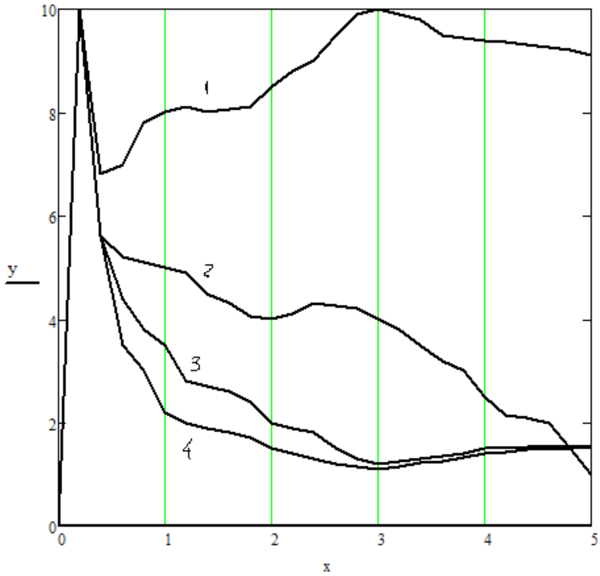
where positive $\lambda_n = \varpi(2n+1)/(2l)$. Comparing the latter expression with (16), we find an obvious difference in generated frequencies.

Pic. 2 shows distribution of response bending moment of a monolithic sleeper lying on the ballast [5]. The positive direction of the y-axis is focused down to bring the graphics in line with the physical meaning and images 1 and 2. The first solid curve, the top curve corresponds to the ideal support sleeper without a defect, in case of evenly distributed load on the roadbed (picture 1). Lower curve 3 corresponds to the calculated torque value. Curve 2 corresponds to the image 2, when 100% of the load is distributed on both ends of sleepers and sleeper's middle is deprived of support.

We use Pic. 2 to show what kind of function $f_0(\chi)$, $f_1(\chi)$ can be selected as initial conditions in model calculations by the formula (13).

Let's explain the meaning of Pic. 3 [5]. To do this, pay attention to the image 2 in Pic. 2, where the middle of a sleeper has no support on ballast. Let's take conditionally moment diagram of the right side of the sleepers support equal to one hundred percent. In Pic. 3 curves 1, 2 and 3, 4 show distribution when the right side of the sleeper has a support of 0, 20, 80 and one hundred per cent, respectively. Zero percent of the support means that the right side of the sleeper does not rest on the roadbed, but hangs on the rail. It can be seen that the difference of boundary conditions at 20 percent or more significantly alters the characteristics that is immediately reflected in the vibration spectrum.

As for the equation problem is linear, in mathematical modeling boundary and initial conditions can be divided between the components in series of tasks, such as (7) and (8). That is to put in the first task (7) zero boundary conditions and to solve the Cauchy problem. In this case, the last two terms in (15) become zero.



Pic. 3. Distribution of dynamic moment on a left part of a sleeper.

Then for the equation (8) initial conditions can be set equal to zero and the boundary problem can be solved by formulas similar to (15), which are given in [11, p. 263].

Further interest to the formulas for solving KGF equation arises when the dimensionless parameter to dissipation function (flow) $K^2 = x_0^2\mu^4/(r^2(a^2-\mu^4))>>1$ is large. Calculations of values K^2 for physical characteristics of corresponding to discussed railway topics in equations (8) and (9) shows that there is just such a case. And then abnormally rapid oscillation of solutions is real [9, p. 26].

The fundamental solution (Riemann function) for the equation

$$\Pi_\mu \Theta(\chi, \tau)+K^2\Theta(\chi, \tau) = \delta(\chi, \tau) \qquad (20)$$

has a form

$$\Theta(\chi,\tau)=\frac{1}{2}J_0(K\sqrt{\tau^2-\chi^2})\theta(\tau^2-\chi^2). \qquad (21)$$

Here j_0 is Bessel function, $\delta(\chi, \tau)$ is Dirac delta function, Θ is theta Heaviside function equal to unity when $\tau^2 > \chi^2$ and zero when $\tau^2 < \chi^2$. In [9] it is shown that the distance between the nearest roots is an indicator of speed of oscillation of functions. With limited values of τ and χ and $\tau^2-\chi^2 \geq const > 0$ this distance has an order of 1/K, which follows from the asymptotic behavior of the Bessel function. Moreover, the oscillation rate increases sharply in the vicinity of the characteristics $\tau = \pm K$.

In the cited paper [9] it was proven that the distance between neighboring roots on the line $\tau = \chi$ is of order $O(1/K^2)$, which indicates an abnormally rapid oscillation. This effect is present in the discussed problem.

5. Exact solution of the problem of rolling wheels on the rail with wavelike wear

In the second model cycle, the researchers considered as a beam railway rail itself with wavelike wear. The dynamics in the coordinate system moving with constant speed is studied. It is noted that another source of vibrations are horizontal and vertical unevenness of left and right rails. In «weldless» track welding places have greater hardness and is worn less than the rail itself, while rebounders are created.

In [7] Dirac delta function is evaluated at a point moving with constant speed, because for construction of a solution in this paper is used the transition to the appropriate coordinate system and then the solution can be expanded in Fourier series. Note that, in practice, it is difficult to provide a uniform movement of the wheel set. See note 1.

As a result, we derived an equation describing the steady vertical rail fluctuations. Designated therein by y(x, t) upward transverse deflection of a rail, which has a bending stiffness E I and is supported by uniform base with hardness u and toughness r, lies at the basis of the formula:

$$E\,I\frac{\partial^4 y(x,t)}{\partial x^4}+m\frac{\partial^2 y(x,t)}{\partial t^2}+ r\frac{\partial y(x,t)}{\partial t}+u\;y(x,t)=F(x,t). \qquad (22)$$

In our case, the function F(x, t) in contrast to [7] contains Dirac delta function, which sets the position of a force applied at the point x = v(t)t moving with variable speed v(t). The solution allows for analysis of motion with variable speed. In the simulation, it is possible to specify different laws of speed change and to analyze their effects on the vibration spectrum. In [7] by m is denoted the linear density of the distributed masses of rail and ballast.

Theorem 4. *Given a mixed problem with boundary conditions for the equation (22). Then the solution has a form*

$$y(x, t) = Z(x, t)\exp(rt/(2m)), \qquad (23)$$

where the function Z(x, t) is an exact solution of the mixed problem for linear standard hyperbolic fourth-order equation

$$m\frac{\partial^2 Z(x,t)}{\partial t^2}+a^2\frac{\partial^4 Z(x,t)}{\partial x^4}+K\;Z(x,t)= =\exp(-rt/(2\rho))\,F(x,t). \qquad (24)$$

Here $a^2 = E\,I$, $K = (4u\rho-r^2)/(4m^2)$.

Proof. *In contrast to [7] the frequency of rail oscillation is determined by solution of the conjugate problem for eigenvalues λ_m for an ordinary differential* *equation of the fourth order $\phi^{(IV)}(x)-\lambda^4\phi(x) = 0$ with appropriate boundary conditions [11, p. 533]. Green function has a form*

$$G(x,\xi,t)= =\sum_{n=1}^{\infty}\left(\frac{\phi_m(x)\phi_n(\xi)}{\|\phi_n\|^2\sqrt{a^2\lambda_n^4+K}}\sin(t\sqrt{a^2\lambda_n^4+K})\right),$$

$$\phi(x)=C_1\exp(-x\lambda)+C_2\exp(x\lambda)+ +C_3\cos(x\lambda)+C_4\sin(x\lambda),$$

$$\|\phi_n\|^2=\int_0^l\phi_n^2\,dx=\frac{l\phi_n^2}{4}+l\left[\phi_n^{''}\right]^2/(4\lambda_n^4)- -l\phi_n^{'}(l)\phi_n^{''''}(l)/(2\lambda_n^4). \qquad (25)$$

Here, the constants C_i, $i = 1,\dots,4$ are determined from boundary conditions. The rate of eigenfunction is calculated by Krylov formula. Hence the oscillation frequency $\sqrt{a^2\lambda_n^4+K}, n=1,2,\dots$ At values of $\rho = 78$ [kg / mm], $u = 34\ 10^6$ [N / m²], $E\,I = 4\ 10^6$ [N / m²], $r = 2\ 10^4$ [N s/ m²], from [7] we get the value of parameter $K>600$. Consequently, in this case, there is an effect of abnormally rapid oscillation.

Conclusions. *In railroad diagnostic centers various local methods and integral method are used, the supporters of which are the authors of this work. If this method had been consistently and universally applied, then a severe accident, a similar to the accident in Moscow metro July 15, 2014, could have been prevented. In most cases the diagnostician is external source of vibrations, and in the present embodiment oscillations with frequencies generated by the moving object are used.*

Analysis of results of mathematical modeling leads to the following conclusions:

a) rail roadbed defects are recorded more or less explicitly in one of the spectra of vertical, longitudinal or transverse vibrations in a wide frequency range;

b) in the vicinity of the main resonances and other defects increase dispersion;

c) taking this into account it is possible to build software and hardware complex for control of the state of railway roadbed with rolling stock.

In our opinion, in general terms, the complex should include:

– Creation of a reference database of spectra of vertical, longitudinal or transverse vibrations with reference to the location of the i-th section using wide / frequency sensors at speed recommended for specialists;

– Selection for each i-th section of reference control points (j = 1,…, m), where the resonance frequency and its characteristics are calculated.

– Calculation of dispersions in the vicinity of the principal resonance for comparison with reference values in excess of critical values repairmen team is sent, specifying the defects by local methods of diagnosis and carrying out the necessary works.

Other specialists may offer additional scenarios and supplement the proposed scheme.

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Article received 30.12.2015, accepted 26.03.2016.