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Modal Analysis of Circular Symmetrical Plates by Means of Generalized Finite Difference Method

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ABSTRACT

In this paper, a simplified modal analysis procedure of circular plates procedures (on polar domains) through generalized (modernized) finite difference method (abbreviated next as – FDM) is developed.

Generally, circular plates are widely used for a plenty of modern civilian and industrial utilities, machine design and many other purposes. They form a spectrum of elements starting with trains' bogies along with engine pistons, dampers and up to slabs and roofs over circular-shaped buildings, train stations and other transportation facilities.

Nowadays, FDM predominates the numerical solutions of partial differential equations (abbreviated next as – PDE) not less than the method of finite elements (abbreviated next as – FEM). This is wide-famous mathematical-discretization method that is economic to compute and simple to code, less regarding to computation tools in hands and how powerful/less powerful they are, since it bases on replacing each derivative by a difference algebraic quotient in a classical formulation. In a sense, a finite difference formulation offers a more direct approach to the numerical solution of the PDE especially in polar coordinates domain problems considering curvilinear dimensions that even FEM does not.

The generalized approach of FDM considers many parameters less regarded by the classical one. Consequently, the use of classical approach negatively affects the accuracy of calculation (convergence to the exact solution values) and the tendency of results, the thing been healed by the generalized approach.

Keywords: *modal analysis, generalized finite difference method (approach), numerical method(s), Eigen model, applied mathematics, circular plate(s).*

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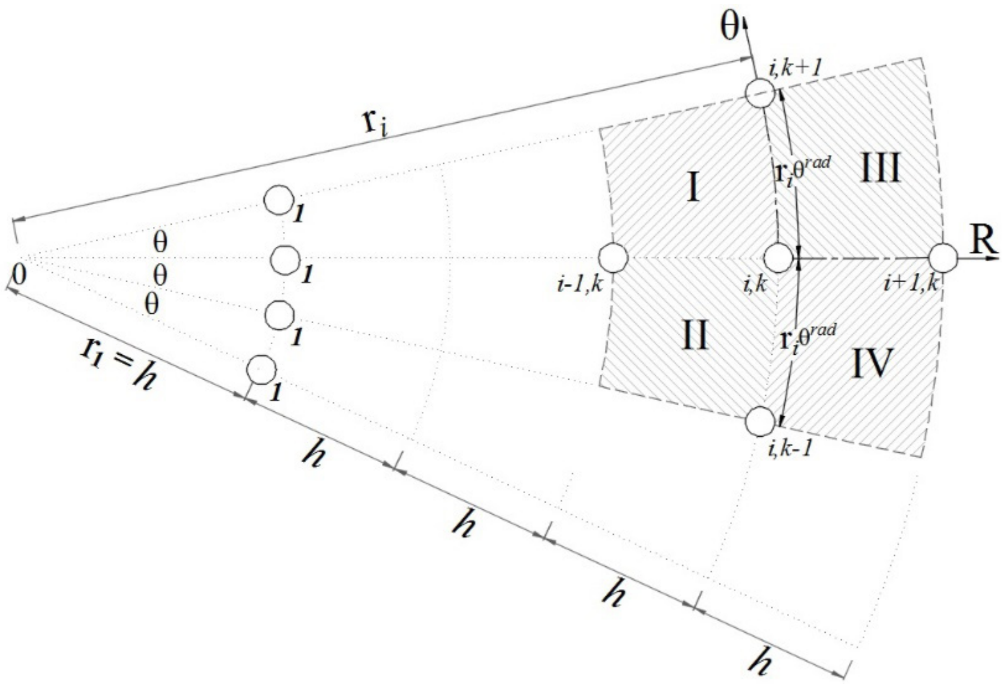


Pic. 1. A sector of plate indicates calculation nodes in different positions.

A free vibration of a plate [2, pp. 1–8; 3, pp. 787–810; 4, pp. 89–110]

The undamped free flexural vibrations of circular plates are basically boundary value problems of the differential mathematical physics. Therefore, the solution in the case of freely vibrating plates can take the form of the following homogeneous differential equation:

$$D\nabla_r^2\nabla_r^2 w(R,\theta,\tau) + \bar{m}\bullet\frac{\partial^2 w(R,\theta,\tau)}{\partial\tau^2} = p_z^* . \quad (1)$$

In most circular plate cases (*especially those of symmetrical nature*), the effects of the rotational inertial forces (P_z^*) are neglected. We can investigate the solution of equation (1) in the form:

$$D\nabla_r^2\nabla_r^2 w(R,\theta,\tau) = -\bar{m}\bullet\frac{\partial^2 w(R,\theta,\tau)}{\partial\tau^2} .$$

In that case the solution of the equation ($D\nabla_r^4 W(R,\theta) = 0$) of biharmonic homogenous nature can logically be extended so that:

$$D\nabla_r^2\nabla_r^2 W(R,\theta) - \lambda\bullet W(R,\theta) = 0 , \quad (2)$$

where $\lambda_n = \frac{\bar{m}\bullet\omega_n^2}{D}$ represents the eigenvectors for each (n).

Generally, equation (2) represents the characteristic equation that describes a free-vibrating circular plate in which each modal vector (λ) complies with a corresponding natural frequency (ω_n).

Numerical approximation of the biharmonic equation by means of generalized FDM

Starting with Laplacian biharmonic operator, formed in polar coordinates as [1, p. 51; 6, pp. 23–27]:

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\bullet\frac{\partial w}{\partial r} + \frac{1}{r^2}\bullet\frac{\partial^2 w}{\partial\theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\bullet\frac{\partial w}{\partial r} + \frac{1}{r^2}\bullet\frac{\partial^2 w}{\partial\theta^2}\right),$$

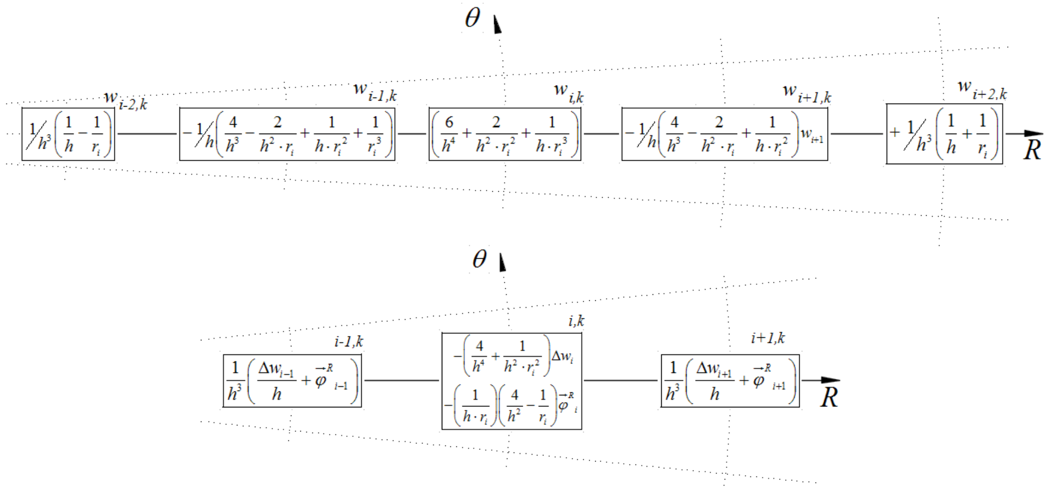
as per [1, p. 51; 5, pp. 119–120] for symmetrical thin circular plates, the previous operator can be simplified so that:

$$\nabla^2\nabla^2 w = \frac{\partial^4 w}{\partial r_i^4} + \frac{2}{r_i}\bullet\frac{\partial^3 w}{\partial r^3} - \frac{1}{r_i^2}\bullet\frac{\partial^2 w}{\partial r^2} + \frac{1}{r_i^3}\bullet\frac{\partial w}{\partial r}.$$

A proposed numerical approximation for the biharmonic solution equation on each node on the integration domain mesh can be stated as per its position so that:

a) For intermediate node, approximation takes the form:

$$\begin{aligned}&\frac{1}{h^3}\left(\frac{1}{h}-\frac{1}{r_i}\right)w_{i-2} - \frac{1}{h}\left(\frac{4}{h^3}-\frac{2}{h^2\bullet r_i}+\frac{1}{h\bullet r_i^2}+\frac{1}{r_i^3}\right)w_{i-1} - \\ &-\left(\frac{4}{h^4}+\frac{1}{h^2\bullet r_i^2}\right)\Delta w_i + \left(\frac{6}{h^4}+\frac{2}{h^2\bullet r_i^2}+\frac{1}{h\bullet r_i^3}\right)w_i - \\ &-\frac{1}{h}\left(\frac{4}{h^3}-\frac{2}{h^2\bullet r_i}+\frac{1}{h\bullet r_i^2}\right)w_{i+1} + \frac{1}{h^3}\left(\frac{1}{h}+\frac{1}{r_i}\right)w_{i+2} + \\ &+\frac{\Delta w_{i-1}+\Delta w_{i+1}}{h^4} + \frac{\vec{\varphi}^R_{i-1}+\vec{\varphi}^R_{i+1}}{h^3} - \left(\frac{1}{h\bullet r_i}\right)\left(\frac{4}{h^2}-\frac{1}{r_i}\right)\vec{\varphi}^R_i = \boxed{\lambda\bullet w_{i,k}} .\end{aligned} \quad (3)$$



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b) For nodes, located near polar mesh's center, the approximation would be:

$$\nabla^2\nabla^2 w_1 = \frac{1}{h^4}\left[(-3)w_0+(7{,}125)w_1-(6)w_{n+1}+(1{,}875)w_{2n+1}\right]=\boxed{\lambda\cdot w_1}\ . \quad (4)$$

c) For central nodes, approximation is shaped as:

$$\nabla^2\nabla^2 w_0 = \frac{1}{h^4}\left[18w_0-24w_1+6w_2\right]=\boxed{\lambda\cdot w_0}\ . \quad (5)$$

d) Boundary conditions are treated as per classical FDM.

Estimation Procedure

• Each calculation node on the domain is supposed to be integrated in a way that corresponds its position as described in the previous section.

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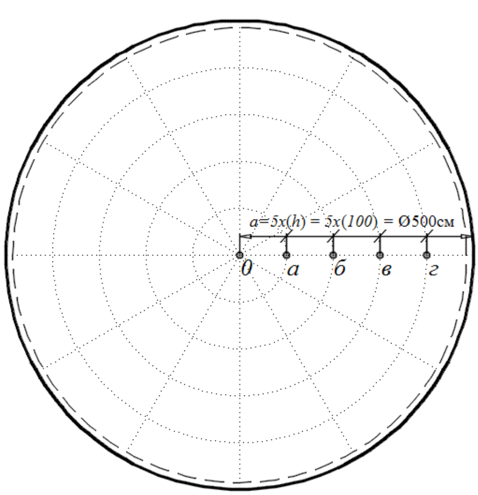


Pic. 2. A meshed plate simply supported along its perimeter.

• A system of linear equations results can be easily tackled via matrix formation of AX = λ • Bx,

where X = {Xi} is a vector matrix which elements represent the amplitudes of free vibration that govern the circular plate;

A = [A_i] is a square matrix, obtained by the generalized finite-difference expressions of the biharmonic operator for out-coming system of equations;

B = [b_i] is a diagonal matrix representing constants that take their places in the system at right-hand side (R.H.S.)

• Multiplying the whole algebraic equations' system by the inverse matrix B^{-1} , we can get (C – λ • I)x = 0; where C = B^{-1} • A & I represents the unity matrix.

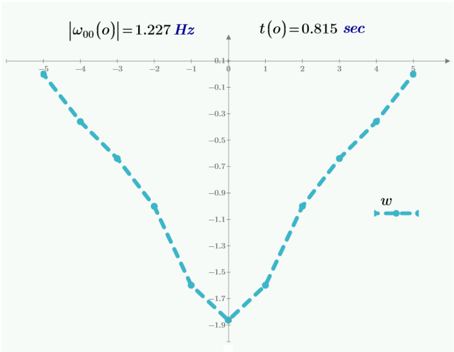
• Except for the trivial solution that equation (C – λ • I)x = 0 gives, a modulus of the parameters set |C – λ • I| implies:

$$|\text{C-}\lambda\text{I}| = \begin{vmatrix} c_{11}-\lambda & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22}-\lambda & c_{23} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33}-\lambda & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn}-\lambda \end{vmatrix} = 0\,,$$

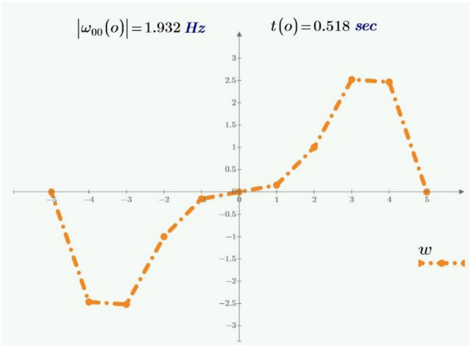
from which we can generate required eigenvectors (modal vectors) corresponding to natural frequencies (ω_n) of considered plates.

Numerical Verification

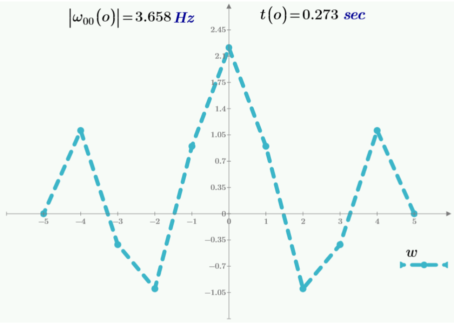
In this part, a simple verification problem had been accurately chosen to express how easy the presented technique can estimate natural frequencies / mode-shapes of a circular plate showing high accuracy compared to other methods applied in common engineering tasks for solving similar problems.



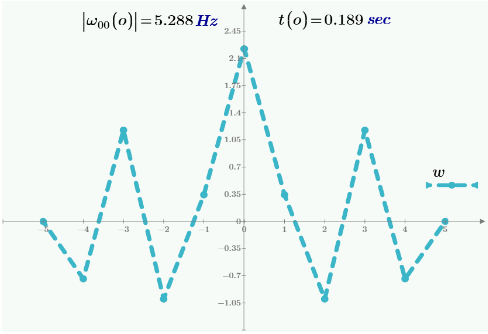
Pic. 3a. 1st mode shape of a plate corresponding to ω_0 = 1,227 Hz.



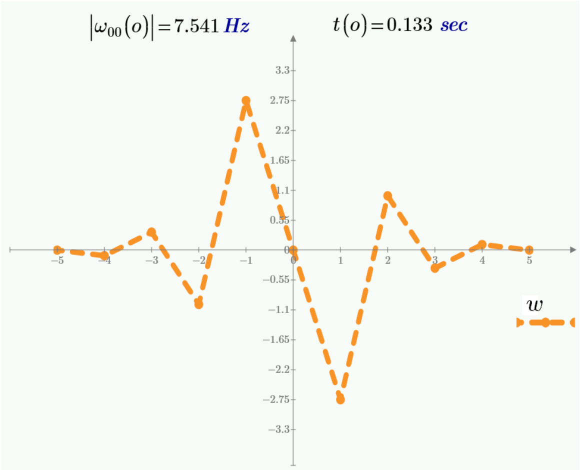
Pic. 3b. 2nd mode shape of a plate corresponding to ω_1 = 1,935 Hz.



Pic. 3c. 3rd mode shape of a plate corresponding to ω_2 = 3,658 Hz.



Pic. 3d. 4th mode shape of a plate corresponding to ω_3 = 5,288 Hz.



Pic. 3e. 5th mode shape of a plate Corresponding to ω_4 = 7,541 Hz.

Let us consider a steel circular plate ($E = 2{,}04 \cdot 10^{-6}$ kg/cm², $\upsilon = 1/3$) of dimensions (a = 500 cm, thickness = 2 cm) with a simply (hinge type) supported perimeter (as that in Pic. 2).

Here, verification consists mainly in evaluation of the precision of eigenvectors (mode shapes) & eigenvalues (natural frequencies) resulted from calculations generated through the suggested approach in comparison with others.

Concerning eigenvalues (*natural frequencies*), a comparison of results has been tabulated as follows (Table 1).

Eigenvectors (*mode shapes*) are readily plotted along a *central x-section* of the investigated plate with the aids of a *Mathcad* sheet as shown in Pics. 3a–e.

Conclusion

The proposed algorithm based on a generalization (*modernization*) of finite difference equations has been clearly verified for circular plates calculations.

Table 1

Results comparison table

n	Eigenvalues (ω_n) in Hz		
	Modernized FDM	Analitical Solution [7, pp. 233–234]	FEM
0	1,227	1,3990	0.98
1	1,932	1,7392	2.07
2	3,658	3,6176	3.31
3	5,288	5,2469	5.11
4	7,541	7,5795	7.83
Error, %	4,92 %	12,22 %	

Table 2

Nomenclature

$w_{i, k}$	The value of deflection (w) at a point-node – of a (i) ordinate on R-axis & (k) ordinate on θ-axis
ω_n	n^{th} natural frequency of the plate
$\frac{d^* u}{dR^*} = u^{*(\prime)} = \partial^* U = \nabla_r^*$	A function (U)'s derivative of a power (*) on axis (R)
$\Delta W_j^{(\to R)}$	Difference in values of deflection before & after node (J) along R-axis
$\varphi_j^{(\to R)}$	Angle of rotation between the horizontal plane & normal to the plate at node (J)
h, $r_j \theta_j$	Mesh dimensions on axes R, θ for a node (J), respectively
$D = \left(\frac{Et^3}{12\left(1-\upsilon^2\right)} \right)$	Cylindrical rigidity. «stiffness» of a circular plate
E, υ, t, a	Elasticity (Young's) modulus, Poisson's ratio and Plate thickness, Plate Radius respectively

Referring back to the numerical verification section, it figured out the core advantage of the suggested procedure, not only limited to precision of the generated results and their convergence to exact solution, compared to the outcomes of the other similar methods, but also a possibility to include new terms that consider discontinuity parameters never considered by the classical FDM, thus allowing to analyze plates with more complex geometry.

Finally, it's not hard to solve problems using numerical techniques especially the above proposed one, as it is a process of handling continuous R.H.S of an original differential equation discretized together with its mathematical derivatives in a system of linear algebraic equations. Raising the solution accuracy through embedding discontinuity parameters, speaks up about the possibility of using the new developed method independently/co-phased with the method of finite elements.

REFERENCES

1. Timoshenko, S. P., Woinowsky-Krieger, S. Theory of Plates and Shells. McGraw-Hill, New York, 1959, second edition, 595 p., Ch. 3, pp. 51–78.

2. Leissa, A. W. Vibration of Plates. National Aeronautics and Space Administration, NASA SP-160, Washington, D.C., 1969, 362 p., Ch. 1, 2, pp. 1–8.

3. Szilard, R. Theories and Applications of Plate Analysis: Classical, Numerical and Engineering Methods. John Wiley & Sons, Inc, New Jersey, 2004, Ch. 14, pp. 787–810.

4. Fried, I. Numerical solution of differential equations. Academic Press, Inc., New York, 1979, 278 p., Ch. 6, pp. 89–110.

5. Den Hartog, J. P. Advanced Strength of Materials. Dover Publications, Inc., New York, 1952, 378 p., Ch. 4, pp. 119–120.

6. Varvak, P. M. Development and application of mesh method to plate calculations: some problems of applied theory of flexibility in finite differences [*Razvitie i prilojenit metoda setok k rashoty plastin: nekotorye zadachi prikladnoi teorii uprugosti v konechnykh raznostiakh*]. 1st part. Kiev, Academy of Sciences of the Ukrainian SSR publ., 1949, 136 p., Ch. 3, pp. 23–27.

7. Handbook on Structural Dynamics [Spravochnik po dinamike sooruzheniy]. Ed. by Korenev, B. G., Rabinovich, I. M. Moscow, Stroyizdat publ., 1972, 511 p., Ch. 8, pp. 233–234. ●