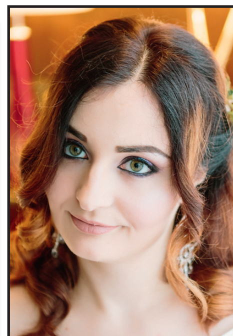
https://doi.org/10.30932/1992-3252-2019-17-3-16-39

Some Methods of QR code Transmission using Steganography





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ABSTRACT

The article presents additional options for development of the previously described method [3] of covertly transmitting a QR code using steganography, which may be required for delivery of information related to the transportation process and other tasks solved in transport. In particular, a new detailed option of application of different mathematical methods used in various scientific fields (for example, multi-grid method for difference approximation of the Dirichlet boundary-value problem for the Poisson equation with a high degree of accuracy) was proposed. An effective iterative formula was constructed for cases of complex sources distribution. The method reduces the number of iterations and the likelihood of an error when restoring the original and allows to create a respective application software.

Keywords: steganography, QR code, effective iterative formula of high accuracy, Poisson's equation, iterative methods, stegacontainer, transport.

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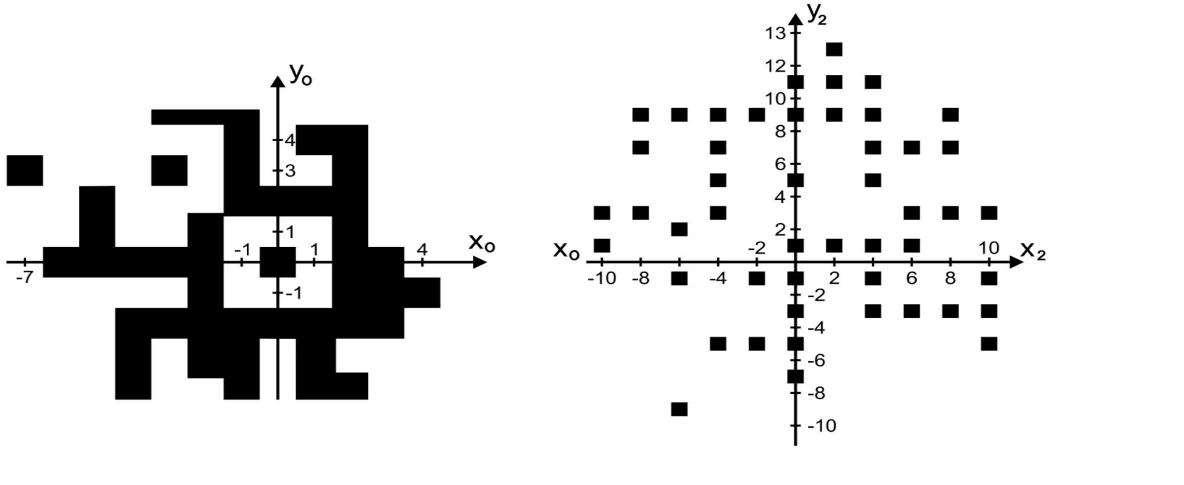
Article received 04.02.2019, accepted 28.05.2019.

For the original Russian text please see p. 16.

Acknowledgements. The authors express gratitude to S. P. Vakulenko, K. A. Volosov, M. A. Basarab, A. L. Balandin, V. G. Danilov, V. V. Pikalov for the attention to the paper, assistance and useful advice.



a) *b)*

Pic. 1a, b. Fragment of a QR code.

Introduction. The scope of modern scientific discussion comprises both issues of using a QR code to solve problems of transmission of information for transportation process purposes, and of methods and algorithms associated with the use of steganography for this purpose. The use of mathematical physics methods allows to increase strength of cryptographic algorithms against the attempts to unauthorizedly acquire information. This will increase safety of transportation and stability of transport operations.

Some important aspects of the problems in transportation of dangerous goods are discussed in [3, p. 14]. In [1, 3], the algorithm of direct and inverse Radon transforms is implemented using a specific example. The same works [1, 3] for the first time proposed to use the steganography for transmission of a QR code.

The [2, 3] for the first time proposed to use a solution of the Dirichlet boundary value problem solution in Poisson equation to transmit a QR code. It is assumed that someone (SO) wants to send information to the addressee (A) encoded in a QR code secretly from an outside observer.

To do this, the given QR code is divided into the same elementary «white» and «black (shaded)» squares and their centers of gravity are calculated.

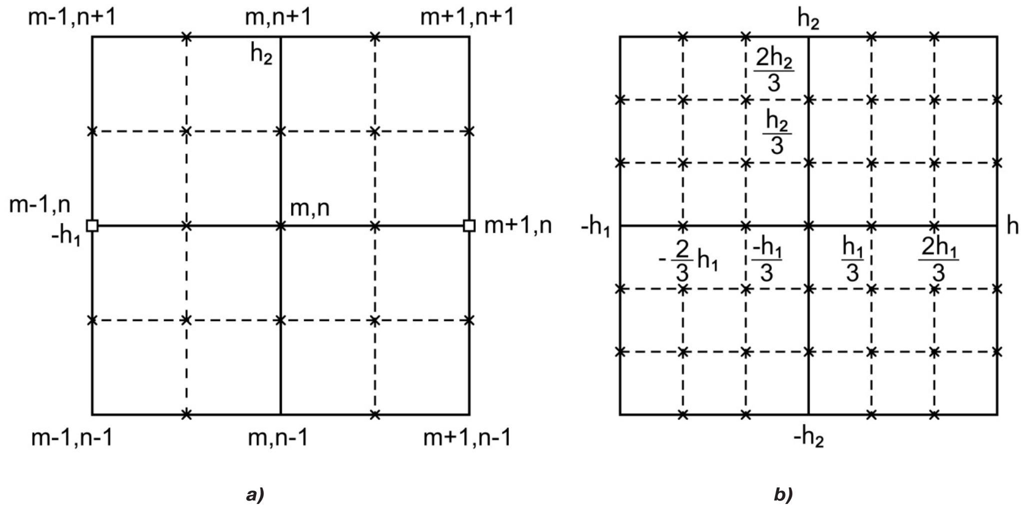
The function of two variables is assigned:

$$\varphi(x,y)=\begin{cases}1, & if\,(x,y)\in\Omega,\\ 0, & if\,(x,y)\notin\Omega\end{cases}, \qquad (1)$$

where Ω is a black (shaded) area (Pic. 1), which can be represented as a combination of a certain set of elementary squares. The masked image is a transformed QR code in which the transmitted information is located. The function (1), which has discontinuities of the first kind at some boundaries of the «black» squares, is calculated.

In [1, 3], methods of preliminary masking and sophistication of a message were proposed, for example, the method used in the theory of fractals, namely, «iteration by linear systems». Accordingly, an intermediate iteration process is organized $x_{s+1} = e_0 + a_0 x_s + b_0 y_s$, $y_{s+1} = f_0 + c_0 x_s + d_0 y_s$, where s = 0,1... iteration number, and x_s, y_s are the coordinates of the centers of gravity of elementary «black» squares that make up the QR code in the Cartesian coordinate system. The origin, the point (0,0) on Ω, the scale and the direction of the axes relative to the graph are known only to SO and are associated with a given QR code. The same applies to the given constants $\{e_0, a_0, b_0, f_0, c_0, d_0\}$ and the selected number of iterations s. Thus, additional keys appear. Note that the structure and geometry during each iteration change significantly, but there is a unique inverse transformation given in [3]. It is possible to transmit any iteration.

The first iteration shown in Pic. 1a, is described in [3, p. 16]. Pic. 1b shows the second iteration of the QR code fragment using linear



Pic. 2a, b. Used grid patterns. The outer rectangle corresponds to a nine-point pattern with a high order of approximation of the Poisson equation in a rectangle [4] and in the «base» grid (6).

systems $a_0 = 1$, $b_0 = -1$, $c_0 = 1$, $d_0 = 1$ $e_0 = 1$, $f_0 = 2$.

The *objective* of this work is to further develop the method for transmitting information previously set forth in [3] using the boundary problem for the Poisson equation by developing an effective iterative formula for a complex system of distributed sources in the problem of steganography and by developing a second program on this basis.

The study used *methods* of applied mathematics, in particular, the method of the Dirichlet boundary value problem solution for the Poisson equation, the Gauss distribution, methods for solving differential equations and systems of linear algebraic equations.

Results.

1. Preliminary remarks

Remark 1. To solve the problem of hidden transmission of QR codes, it is proposed to biuld two software programs in parallel. The first program describes the procedure for calculating the solution of a boundary value problem with zero boundary conditions in a rectangle for the Poisson equation, which is suitable for a large family of QR codes. All keys (required constant values) are computed or specified. The second program restores the original using the image separated from the «stegacontainer».

In the model version described in this work three objects are considered: the original «O», the image obtained after the integral transformation «I», the restored original «RO». The restored original «RO» may differ from «O» by an error arising due to noise and to solution of the inverse ill-posed problem.

Remark 2. Note that in the calculations, the integer coordinates of the centers of gravity of elementary squares for formula (5) were specified in the second program manually.

Definition. The masking function will be called the function $\chi(x, y)$, which can specify the coordinates of «black» squares that did not exist in the given QR code and that supplement their set.

Let's build the amount:

$$f(x, y) = \phi\chi(x, y) + \chi(x, y). \qquad (2)$$

Images on the projection plane of the graph of the function (2) are considered analogously to the example in Pic. 1. Similarly to (1), we assume that at the points (x, y), belonging to the «black» squares, the function (2) takes the value of one, and the function is zero at the points (x, y) belonging to «white» squares. At the stage of difference approximation of the problem, we use the «twin» of the function f(x, y) in the form of a discontinuous function

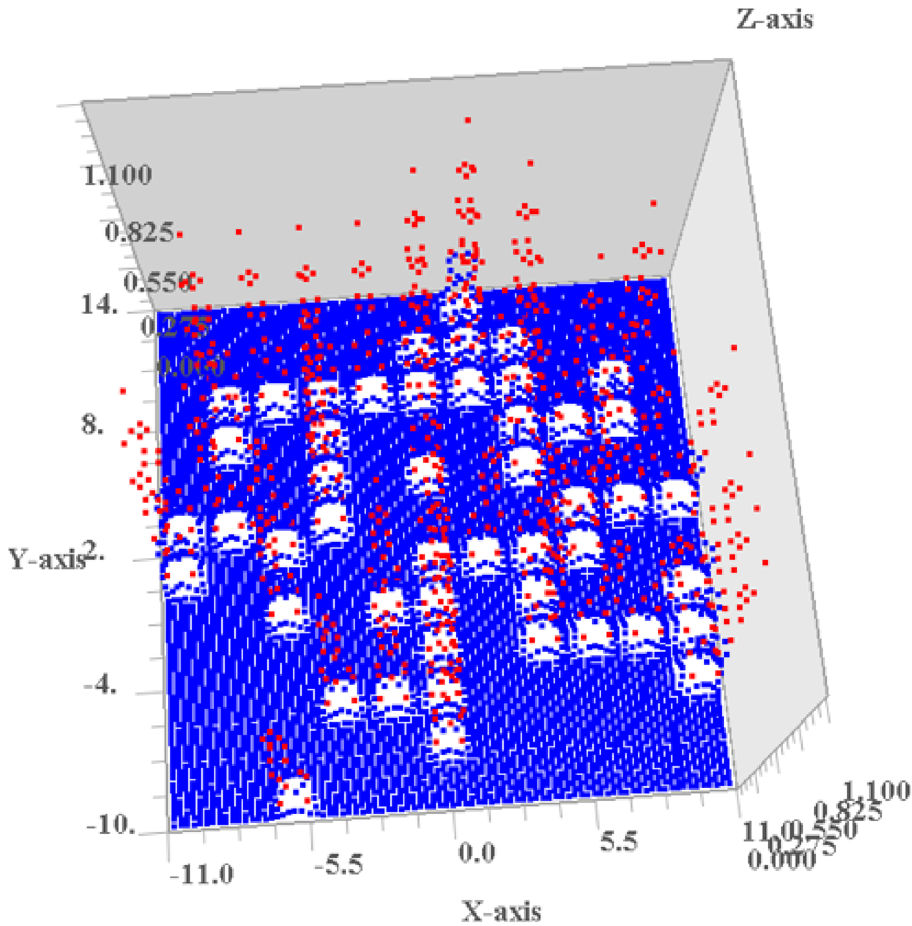
$$N(x, y) = \begin{cases} 1, & if\ (x, y) \in \Omega, \\ 0, & if\ (x, y) \notin \Omega \end{cases}. \qquad (3)$$

We will call (3) an «indicator» function.

Its role is described below in Remark 3.

2. The method of information transmission using the boundary value problem for the Poisson equation

We will place the graph corresponding to function (2) on the plane inside the contour, which represents a rectangle of such a minimum



Pic. 3. The function of the right side of the problem (4), calculated by the formula (5) for the source shown in Pic. 1b.

area with sides parallel to the axes of the Cartesian coordinate system, that all «black» (shaded) squares are inside it. The case when this rectangle turns out to be a square, is rare and simpler, it was briefly considered in [3, p. 19]. Then it is possible to do with a uniform grid with a constant step h. Typically, in such a formation a QR code generates a rectangle on a plane. If we use a step obtained by splitting the minimum side, then in order to reduce computations in the whole area, it is obviously necessary from practical point of view to introduce the second somewhat larger step in the direction of the long side of the rectangle. The sides of an elementary black shaded rectangle should be related as rational numbers, for example, 1 : 2. By this, we reduce the number of calculations, but the «black» squares are transformed into rectangles, and the area occupied by the formulas increases by about three times.

So, in many cases, the quadrilateral contour of the minimum area, covering all shaded elementary squares, in the graphic image of the function f(x, y) (3) turns out to be a rectangle (Pic. 1). Let's denote the inner area by Ω_0. For certainty of b and for brief formulas, we assume $x_{max} = a > 0$, $y_{max} = d > 0$, $x_{min} = b < 0$, $y_{min} = c < 0$.

We denote by $m_1 = |x_{max} - x_{min}|$, $m_2 = |y_{max} - y_{min}|$ the lengths of the sides of the contour rectangle. Let's consider the internal Dirichlet boundary value problem for the Poisson equation with homogeneous boundary conditions:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y), \qquad (4)$$

a < x < b, c < y < d, u(a, y) = u(b, y) = u(x, c) = u(x, d) = 0.

The entire area is divided into elementary «white» and «black» rectangles by the first «large» two-dimensional natural grid.

At the first stage of the transformation of the problem (4), we replace the function with discontinuities of the first kind (4) with a continuous, infinitely differentiable function, namely with the sum of two-dimensional Gaussians. Then the conditions of existence and uniqueness of the theorem for the solution of problem (3)–(5) are satisfied [4, p. 137]. Here we have introduced the following notation: V is the set of pairs of centers of Cartesian coordinates on the plane of «black» squares inside Ω, and W is the set of pairs of coordinates of the centers of gravity on the plane of all «white» and «black» squares inside Ω. At the second stage of the difference approximation of the problem (4) in the program we use the function:

$$f_{mn} = \sum_{\nu,\delta=1}^{m_1,m_2} N_{\nu,\delta} \exp\left(-\left(\frac{x_m - x_\nu}{\eta \cdot h_{11}}\right)^2 - \left(\frac{y_n - x_\delta}{\eta \cdot h_{22}}\right)^2\right),$$

$$\nu = \overline{1,m_1}, \delta = \overline{1,m_2}, \nu, \delta, m, n \in N,\ b - a = h_{11} \cdot m_1 = \quad (5)$$

$$= h_1 \cdot n_1;\ d - c = h_{22} \cdot m_2 = h_2 \cdot n_2, \nu = \overline{1,m_1},$$

$$\delta = \overline{1,m_2}, m = \overline{0,n_1}, n = \overline{0,n_2}.$$

Here (h_{11}, h_{22}) is a coarse-grid vector; the steps of the coarse grid are equal to the sides of the large black (white) cell for specifying the QR code, (h_1, h_2) is the minimum grid step vector for solving the Poisson equation:

$x_m = a + mh_1$, $y_n = c + nh_2$, m = 0,..., n_1, n = 0,..., n_2, $x_\nu = a + \nu h_{11}$, $y_\delta = c + \delta h_{22}$,

ν = 1,..., m_1, δ = 1,..., m_2, (ν, δ) ∈ V, (x_m, y_n) ∈ W.

Thus, a second, two-dimensional uniform «base» grid [9] (Pic. 2a) was introduced with steps:

$$a < x < b, c < y < d,$$

$$\omega_{n_1,n_2} = \left\{ \begin{array}{l} x_m = a + mh_1,\ y_n = c + nh_2, m = \overline{0,n_1}, \\ n = \overline{0,n_2}, h_1 = \frac{m_1}{n_1} h_{11} = \frac{b-a}{n_1}, h_2 = \frac{m_2}{n_2} h_{22} = \frac{d-c}{n_2} \end{array} \right\}. \quad (6)$$

The lengths of the sides of an elementary «white» and «black» rectangle are in general considered by the program as multiples of the smaller steps of the «base» grid h_1, h_2. The centers of «black» squares (rectangles in the general case) (x_δ, y_ν) are determined by formula (5), and in the example in this paper, the coefficient responsible for variance in functions (5) is chosen as η = 0.5. This choice is not the single one.

Remark 3. Note that n_1, n_2, m_1, m_2 are positive integers, program parameters. The program uses for technical purposes the differential approximation of the «indicator» function (3) $N_{\nu,\sigma}$ on the grid (6):

$$\nu = \overline{1,m_1}, \delta = \overline{1,m_2}, N_{\nu,\delta} = \begin{cases} 1, (\nu,\delta) \in N_1^2 \\ 0, (\nu,\delta) \in N_2^2 \end{cases}, \quad (7)$$

where N_1^2 is a two-dimensional set of pairs of node numbers – the centers of «black» rectangles. Similarly: N_2^2 is a two-dimensional set of pairs of nodes – the centers of «white» rectangles, $N_1^2 U N_2^2 = W$.

Function (5) defines the complex structure of distributed sources. The center of a Gaussian is a point on the plane in which the local maximum of the function is located, which, similar to the Gaussian distribution, coincides with the center of gravity of the «black» square.

Firstly, (5) well approximates and smooths the boundary of the discontinuity between the «white» and «black» squares (in general, rectangles), which has existed before. The «indicator» function performs the erasure of the «tails» of the Gauss distribution, if they «got in» the «white» squares (in general, rectangles). Secondly, it makes it easier to check stability of the work of both programs (see Remark 1) not only in the example under consideration (Pic. 1), but also for any other representative of the family of functions (2) generated by another QR code.

Remark 4. There is an exact solution to the problem (4), (5), expressed in terms of the Green's function [5, p. 126], which, unfortunately, is only of theoretical interest. In practice, it generates the sum of rapidly oscillating terms of a series with slowly decreasing coefficients, while it is necessary to calculate the integrals of them.

Remark 5. Based on the need to present the material in an understandable and simple form using the example of simple formulas, we skip a number of steps. For practical application of the proposed method, more complex formulas can be found in [9]. At the same time, we leave markers for the process of restoring formulas in the form of different steps, and write some simple formulas for two different steps. The most cumbersome formulas are given for the simplified case $h_1 = h_2 = h$, whereas separate formulas for different steps are three times more voluminous and have an approximation error lower by 2 orders of magnitude [4], [9].

$$
\begin{gathered}
f(x_{m-1},y_{n-1})=f_{-1,-1},\quad f(x_{m+1},y_{n-1})=f_{1,-1},\ f(x_{m-1},y_{n+1})=f_{-1,1},\ f(x_{m+1},y_{n+1})=f_{1,1},\quad (Z_3)\\
f(x_{m+1},y_n)=f_{1,0},\quad f(x_{m-1},y_n)=f_{-1,0},\quad f(x_m,y_{n+1})=f_{0,1},\quad f(x_m,y_{n-1})=f_{0,-1},\quad (K_1)\\
f(x_{m-1/2},y_n)=f_{-1/2,0},\ f(x_{m+1/2},y_n)=f_{1/2,0},\ f(x_m,y_{n+1/2})=f_{0,1/2},\ f(x_m,y_{n-1/2})=f_{0,-1/2},\ (K_2)\\
f(x_{m+1/3},y_n)=f_{1/3,0},\ f(x_{m-1/3},y_n)=f_{-1/3,0},\ f(x_m,y_{n+1/3})=f_{0,1/3},\ f(x_m,y_{n-1/3})=f_{0,-1/3},\ (Q_1)\\
f(x_{m+2/3},y_n)=f_{2/3,0},\ f(x_{m-2/3},y_n)=f_{-2/3,0},\ f(x_m,y_{n+2/3})=f_{0,2/3},\ f(x_m,y_{n-2/3})=f_{0,-2/3},\ (Q_2)\\
f(x_{m-1/2},y_{n-1/2})=f_{-1/2,-1/2},\quad f(x_{m-1/2},y_{n+1/2})=f_{-1/2,1/2},\quad f(x_{m+1/2},y_{n-1/2})=f_{1/2,-1/2},\\
f(x_{m+1/2},y_{n+1/2})=f_{1/2,1/2},\ (Z_1)\\
f(x_{m-1/2},y_{n-1})=f_{-1/2,-1},\ f(x_{m-1/2},y_{n+1})=f_{-1/2,1},\quad f(x_{m+1/2},y_{n-1})=f_{1/2,-1},\\
f(x_{m+1/2},y_{n+1})=f_{1/2,1},\ f(x_{m-1},y_{n-1/2})=f_{-1,-1/2},\quad f(x_{m-1},y_{n+1/2})=f_{-1,1/2},\\
f(x_{m+1},y_{n-1/2})=f_{1,-1/2},\ f(x_{m+1},y_{n+1/2})=f_{1,1/2},\quad (Z_2)
\end{gathered}
\qquad (12)
$$

3. Building an effective iterative formula for a complex system of distributed sources in the problem of steganography

To test the program, let's build an example:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=2\sin(x)+\sin(y),\ 0<x<\pi,\ 0<y<\pi,\\
u(0,y)=u(\pi,y)=0,\quad u(x,0)=u(x,\pi)=0,
\end{cases}
\qquad (8)
$$

with the exact solution written through elementary functions:

$$
\begin{gathered}
u(x,y)=2\sin(x)\left(ch(y)-1+\left(\frac{1-ch(\pi)}{sh(\pi)}\right)sh(y)\right)+\\
+\sin(y)\left(\left(\frac{1-ch(\pi)}{sh(\pi)}\right)sh(x)+ch(x)-1\right).
\end{gathered}
\qquad (9)
$$

In this example, the right-hand side in (8) is a continuously differentiable function inside the domain, and the solution satisfies zero boundary conditions.

We will compare the numerical solution of problem (4)–(5) proposed in this work with the solution (9) of problem (8) by the number of iterations and accuracy, if instead of the right part we use the well-known analytical function (8) and its exact solution (9) instead of the algorithmically specified QR code using formula (5), since the analytical solution for the right side of the Poisson equation, as shown in Pic. 3, is unknown.

Derived relation 1. For a continuously differentiable function f(x, y) in the domain Ω_0 from (4), we have the derived relations:

$$
\begin{gathered}
\frac{\partial^2 u}{\partial y^2}=f(x,y)-\frac{\partial^2 u}{\partial x^2},\quad \frac{\partial^4 u}{\partial y^4}=\frac{\partial^2 f(x,y)}{\partial y^2}-\frac{\partial^4 u}{\partial x^2\partial y^2},\\
\frac{\partial^6 u}{\partial y^6}=\frac{\partial^4 f}{\partial y^4}-\frac{\partial^6 u}{\partial x^2\partial y^4},\quad \frac{\partial^6 u}{\partial x^4\partial y^2}=\frac{\partial^4 f}{\partial x^2\partial y^2}-\frac{\partial^6 u}{\partial x^2\partial y^4},\ etc.
\end{gathered}
\qquad (10)
$$

Formulas (10) follow from (4).

For a brief record, the notation is introduced:

$$
\begin{gathered}
u(x_{m-1},y_{n-1})=u_{-h_1,-h_2},\ u(x_{m+1},y_{n-1})=u_{h_1,-h_2},\\
u(x_{m-1},y_{n+1})=u_{-h_1,h_2},\ u(x_{m+1},y_{n+1})=u_{h_1,h_2},\\
u(x_m,y_n)=u_{0,0},\quad u(x_{m-1},y_n)=u_{-h_1,0},\\
u(x_{m+1},y_n)=u_{h_1,0},\quad u(x_m,y_{n-1})=u_{0,-h_2},\\
u(x_m,y_{n+1})=u_{0,h_2}.
\end{gathered}
\qquad (11)
$$

Similarly to (11) we introduce the notation (12).

Theorem 1. An effective iterative formula for a complex system of distributed sources in the problem (4)–(5) in the case $h_1=h_2=h$ has the form:

$$
\begin{gathered}
u^{k+1}(x_m,x_n)\equiv u^{k+1}_{m,n}\equiv u^{k+1}_{0,0}=\frac{F_2}{5}+\frac{F_1}{20}-\\
-h^2\left[\begin{gathered}\frac{3}{10}f_{0,0}+\frac{1}{40}\left(-\frac{K_1}{3}+\frac{16}{3}K_2-20f_{00}\right)+\\
+\frac{1}{1200}\left(1512f_{00}-\frac{1053Q_1}{2}+162Q_2-\frac{27K_1}{2}\right)\end{gathered}\right]-\\
-\frac{h^2}{300}\left[\begin{gathered}100f_{0,0}-\frac{160K_2}{3}+\left(\frac{16}{3}\right)^2Z_1-\\
-\frac{16}{9}Z_2+\frac{Z_3}{9}+\frac{10K_1}{3}\end{gathered}\right]+O(h^6).
\end{gathered}
\qquad (13)
$$

The following notations are introduced:

$$
\begin{gathered}
F_1=u_{-h,-h}+u_{h,-h}+u_{-h,h}+u_{h,h},\\
F_2=u_{-h,0}+u_{h,0}+u_{0,-h}+u_{0,h},\quad f(x_m,y_n)=f_{0,0},
\end{gathered}
\qquad (14)
$$

and

$$
\begin{gathered}
K_1=f_{1,0}+f_{-1,0}+f_{0,1}+f_{0,-1},\\
K_2=f_{-1/2,0}+f_{1/2,0}+f_{0,1/2}+f_{0,-1/2},\\
Q_1=f_{1/3,0}+f_{-1/3,0}+f_{0,1/3}+f_{0,-1/3},\\
Q_2=f_{2/3,0}+f_{-2/3,0}+f_{0,2/3}+f_{0,-2/3},\\
Z_1=f_{-1/2,-1/2}+f_{-1/2,1/2}+f_{1/2,-1/2},+f_{1/2,1/2},\\
Z_2=f_{-1/2,-1}+f_{-1/2,1}+f_{1/2,-1}+f_{1/2,1}+\\
+f_{-1,-1/2}+f_{-1,1/2}+f_{1,-1/2}+f_{1,1/2},\\
Z_3=f_{-1,-1}+f_{1,-1}+f_{-1,1}+f_{1,1},
\end{gathered}
\qquad (15)
$$

which are simplified. For example, one should read: $f_{-1,-1}=f(x_{m-1},y_{n-1})$, v$f_{1,-1}=f(x_{m+1},y_{n-1})$, $u_{0,0}=u(x_m,y_n)$...and so on (see Pic. 2). Thus, the counting inside the pattern goes from the center point $(x_m,y_n)=(0,0)$.

Proof. We choose a nine-point pattern with a high order of approximation of the Poisson equation in a rectangle (Pic. 2). In [3], a formula was obtained approximating the sum of four function values at nodes with vertices of a rectangle not lying on the coordinate axes (Pic. 2a) $(-h_1,-h_2)$, $(-h_1,h_2)$, $(h_1,-h_2)$, (h_1,h_2).

We rewrite it for equal steps $h_1 = h_2 = h$, using the decomposition into a Taylor series for small values of h (Pic. 2a):

$$u_{-h,-h}+u_{h,-h}+u_{-h,h}+u_{h,h}=4u_{0,0}+ \left.+\begin{pmatrix} 2h^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)+ \\ +\dfrac{h^4}{6}\left(\dfrac{\partial^4 u}{\partial x^4}+6\dfrac{\partial^4 u}{\partial x^2\partial y^2}+\dfrac{\partial^4 u}{\partial y^4}\right)\end{pmatrix}\right|_{\substack{x=x_m\\ y=y_n}}+O(h^6). \quad (16)$$

If the vertices of the rhombus are located in nodes lying on the coordinate axes $(-h_1, 0)$, $(0, h_2)$, $(0, h_1)$, $(h_2, 0)$ (Pic. 2a), then for equal steps $h_1 = h_2 = h$, we get a Taylor expansion:

$$u_{-h_1,0}+u_{h_1,0}+u_{0,-h_2}+u_{0,h_2}= =4u_{0,0}+\left.\begin{pmatrix} h^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)+ \\ +\dfrac{h^4}{12}\left(\dfrac{\partial^4 u}{\partial x^4}+\dfrac{\partial^4 u}{\partial y^4}\right)\end{pmatrix}\right|_{\substack{x=x_m\\ y=y_n}}+O(h^6). \quad (17)$$

Following [4], we approximate the Laplace operator with a linear quadrature formula:

$$\Delta u \approx \frac{1}{h^2}\begin{pmatrix} C_0 u_{0,0}+C_1\left(u_{-h,0}+u_{h,0}+u_{0,h}+u_{0,-h}\right) \\ +C_2\left(u_{-h,-h}+u_{h,-h}+u_{-h,h}+u_{h,h}\right)\end{pmatrix}, \quad (18)$$

where constants are to be defined.

Lemma 1.

For a continuously differentiable function f(x, y) from $C^\infty(R^2)$ in the problem (4)–(5), the constants and the residual of formula (18) have the form $C_1 = 2/3$, $C_2 = 1/6$, $C_0 = -10/3$,

$$R(f)=\frac{h^2}{12}\Delta f(x_m,y_n)+ +\frac{h^4}{360}\left(\frac{\partial^4 f(x_m,y_n)}{\partial x^4}+\frac{\partial^4 f(x_m,y_n)}{\partial y^4}\right)+ +\frac{h^4}{90}\frac{\partial^4 f(x_m,y_n)}{\partial x^2\partial y^2}+O\left(h^6\right). \quad (19)$$

Proof will be given for a simplified case $h_1 = h_2 = h$.

In [4; 6, p. 23] it is proved that only the terms in the fourth and sixth degree of step h can be expressed in terms of the partial derivatives of the function f(x, y) at the central point of the pattern (Pic. 3).

Substituting in (16) the blanks of formulas (14), (15) and using the derived relation (10), we obtain three algebraic equations $C_0 + 4C_1 + 4C_2 = 0$, $C_1 + 2C_2 = 1$, $C_1 - 4C_2 = 0$. These are necessary conditions for equality to zero of the terms, grouped in zero and second order of degree of the step h. This implies $C_1 = -1 - C_0/2$, $C_2 = 1 + C_0/4$.

The necessary condition for the fact that in bracket (19), where the summands are collected in the fourth order in step h, there are no partial derivatives of the function u(x, y) and the values of only the partial derivatives f(x, y) remain, is the equation $C_0 = -10/3$. Then $C_1 = 2/3$, $C_2 = 1/6$.

The proof of Lemma 1 is complete.

We introduce the third smaller uniform grid for the difference approximation of the second partial derivatives of the function f(x, y) by five nodes along each axis. Its pattern is shown in Pic. 2a Additional nodes are marked with crosses.

We introduce a fourth, smaller uniform grid for the difference approximation of the fourth partial derivatives of the function f(x, y) by seven nodes along each axis. Its pattern is shown in Pic. 2b.

Finally, we definitively rewrite the formula (18), where we consider the Poisson equation in the problem (4)–(5):

$$\frac{1}{h^2}\left(-\frac{10}{3}u_{0,0}+\frac{2}{3}F_2+\frac{1}{6}F_1\right)= =f_{0,0}+\left.\left(h^2\frac{\Delta f}{12}+\frac{h^4}{360}\left(\frac{\partial^4 f}{\partial x^4}+\frac{\partial^4 f}{\partial y^4}\right)+\frac{h^4}{90}\frac{\partial^4 f}{\partial x^2\partial y^2}\right)\right|_{\substack{x=x_m\\ y=y_n}}+ +O\left(h^6\right)=0. \quad (20)$$

Here the notation is defined in (11)–(15). The residual (19) is added to the right-hand side of (20), which gives an amendment to the solution, and we will take it into account using iterative numerical methods.

Expressing from (20) the central nodal value $u_{0,0}$, we build a blank of the explicit formula for a simple iteration method:

$$u^{\kappa+1}_{0,0}=\frac{F^k_{\ 2}}{5}+\frac{F^k_{\ 1}}{20}-\frac{3h^2}{10}f^k_{\ 0,0}- -h^2\left.\begin{pmatrix} (h^2\dfrac{\Delta f(x,y)}{40}+ \\ +\dfrac{h^4}{1200}\begin{pmatrix}\dfrac{\partial^4 f(x,y)}{\partial x^4}+ \\ +\dfrac{\partial^4 f(x,y)}{\partial y^4}\end{pmatrix}+\dfrac{h^4}{300}\dfrac{\partial^4 f(x,y)}{\partial x^2\partial y^2})\end{pmatrix}^k\right|_{\substack{x=x_m\\ y=y_n}}+O\left(h^6\right). \quad (21)$$

Here k is the iteration number. In the right part, all terms are calculated at the k-th step of the iteration process. When k = 0, the initial value is selected as a smooth, continuously differentiable function. The formulas for calculating the right-hand side of (21) are derived below.

Since the right side of the Poisson equation in the problem of steganography is given numerically in the form of (5), we will use difference operators for the partial derivatives in (21) from [6, p. 22; 7, p. 33]. We use for this the third and fourth grids with fractional steps for the patterns described above.

For further proving Theorem 1, we need three more auxiliary statements.

Lemma 2. The Laplace operator with $O(h^4)$ accuracy in the form of a quadrature formula in the case $h_1 = h_2 = h$ can be represented as:

$$\Delta f(x_m, y_n) \approx$$

$$\approx \left(-\frac{K_1}{3} + \frac{16K_2}{3} - 20 f_{0,0}\right) + O\left(h^4\right), \qquad (22)$$

where K_1, K_2, $f_{0,0}$ are defined in (11), (15).

Proof. The linear Laplace operator is symmetric with respect to variables x, y:

$$\Delta f = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}. \qquad (23)$$

Then it is sufficient to approximate the second derivative with the specified accuracy.

Remark 6. Now we consider, temporarily (to derive a formula and to show that these arguments will be generalized for a function of two variables f(x, y)) a continuosly differentiable function of one variable with the same identifier f(x). We choose an additional (temporary, for derivation of the formula) local coordinate system in the pattern with zero at the point x_m, y_n.

A formula is known for three equidistant nodes $f_{xx} = (f(-h) + f(h) - 2f(0))/h^2 = (f_{-1} + f_1 - 2 f_0)/h^2 + O(h^2)$, which has an accuracy of $O(h^2)$ [6, p. 22; 7, p. 33] on the x-axis (Pic. 2a). We choose five equidistant nodes within the segment [-h, h], symmetrically located relative to the grid origin:

$$f''_{xx}(0) \approx \frac{1}{h^2}\begin{pmatrix} B_0 f(0) + B_1\left(f(-\frac{h}{2}) + f(\frac{h}{2})\right) + \\ + B_2\left(f(-h) + f(h)\right)\end{pmatrix} \equiv \qquad (24)$$

$$\frac{1}{h^2}\left(B_0 f_0 + B_1\left(f_{-1/2} + f_{1/2}\right) + B_2\left(f_{-1} + f_1\right)\right).$$

Here it is necessary to define constants.

Using the method of indefinite coefficients, we select weight coefficients in the formula (24) so that it has the maximum algebraic order of error, following [8, p. 40].

Remark 7. We assume that in a small neighborhood of zero the function f(x) behaves like a power function. We write the function, calculate the derivative and construct algebraic equations using formula (24):

f [illegible] $\quad f''_{xx} \quad =$

1) $\left(B_0 + B_1(1+1) + B_2(1+1)\right) = 0;$

$B_0 + 2B_1 + 2B_2 = 0;$

$f(x) = x; \quad f''_{xx}(0) = 0;$

2) $\left(B_0 0 + B_1\left(-\frac{h}{2} + \frac{h}{2}\right) + B_2(-h+h)\right) = 0;$

$0 \equiv 0;$

$f(x) = x^2; f_{xx}(0) = 2,$

3) $B_0 0^2 + B_1\left(\left(-\frac{h}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right) + B_2\left((-h)^2 + (h)^2\right) = 0;$

$\frac{2B_1}{4} + 2B_2 = 2;$

f [illegible] $x^3 \quad f''_{xx} \quad = \left. x\right|_{x=0} =$

4) $B_0 0^3 + B_1\left(\left(-\frac{h}{2}\right)^3 + \left(\frac{h}{2}\right)^3\right) +$

$+ B_2\left((-h)^3 + (h)^3\right) = 0; \quad 0 \equiv 0;$

$f(x) = x^4; f_{xx}(0) = 12x^2\big|_{x=0} = 0;$

5) $\left(B_0 0^4 + B_1\left(\left(-\frac{h}{2}\right)^4 + \left(\frac{h}{2}\right)^4\right) + B_2\left((-h)^4 + h^4\right)\right) = 0;$

$\frac{2B_1}{16} + 2B_2 = 0;$

6) $f(x) = x^5; \quad f_{xx}(0) = 20x^3\big|_{x=0} = 0; \quad 0 \equiv 0.$

We obtain SLAE (a system of linear algebraic equations) and its solution:

$B_0 + 2b_1 + 2b_2 = 0$; $B_1 + 4B_2 = 4$; $B_1 + 16B_2 = 0$; $B_0 = -10$; $B_1 = 16/3$; $B_2 = -1/3$.

Let us substitute the results of the calculations in (24) and obtain the rule by which the second-order finite-difference operator A acts on a function of the variable x:

$$f_{xx}(0) \approx A \circ f\big|_{x=0} =$$

$$= \frac{1}{h^2}\begin{pmatrix} -10 f_0 + \frac{16}{3}\left(f_{-1/2} + f_{1/2}\right) - \\ -\frac{1}{3}\left(f_{-1} + f_1\right)\end{pmatrix} + O\left(h^4\right). \qquad (25)$$

In formulating the formula (25), considering symmetry, six conditions were used regarding coefficients, exact for power polynomials of one variable: 1, x, x^2, x^3, x^4, x^5. The left and right parts of the formula (23), the second derivative and quadrature formula (linear with respect to the nodal values of f_i) and fixed weight coefficients B_i are linear functionals. Then their difference, equal to the residual of approximation of the formula (23), is also a linear functional. Therefore, if the residual of formula (23) is zero for the indicated power coordinate functions 1, x, x^2, x^3, x^4, then due to linearity of the residual, the formula (23) is exact for all algebraic polynomials of degree not higher than 5. That is, it is proved that the error order of the numerator of the right side of (23) is 6, that is, $O(h^6)$. Therefore, the error of the second derivative in (23) is $O(h^4) = O(h^6)/h^2$.

By symmetry, we construct a second finite difference operator of second order S, which by the same rule acts on a function of the variable y by virtue of symmetry (21).

For the function of two variables f(x, y) for the Laplace operator, in view of (21), (23), we obtain:

$$\Delta f \approx A \circ f(x_m, y_n) + S \circ f(x_m, y_n) = \\ = \frac{1}{h_1^2}\left(-10f_{0,0} + \frac{16}{3}\left(f_{-1/2,0} + f_{1/2,0}\right) - \frac{1}{3}\left(f_{-1,0} + f_{1,0}\right)\right) + \\ + \frac{1}{h_2^2}\begin{pmatrix} -10f_{0,0} + \frac{16}{3}\left(f_{0,-1/2} + f_{0,1/2}\right) - \\ = \frac{1}{3}\left(f_{0,-1} + f_{0,1}\right)\end{pmatrix} + O\left(h_1^4 + h_2^4\right). \quad (26)$$

For equal steps $h_1 = h_2 = h$, the formula (24) transforms into the formula:

$$\Delta f \approx \frac{1}{h^2}\begin{pmatrix} -\frac{1}{3}\left(f_{1,0} + f_{-1,0} + f_{0,1} + f_{0,-1}\right) + \\ + \frac{16}{3}\left(f_{\frac{1}{2},0} + f_{-\frac{1}{2},0} + f_{0,\frac{1}{2}} + f_{0,-\frac{1}{2}}\right) - \\ -20f_{0,0}\end{pmatrix} + O\left(h^4\right), \quad (27)$$

that is (22).

The proof of Lemma 2 is complete.

Lemma 3. The sum of the fourth partial derivatives in (17) with the accuracy $O(h^4)$ in the form of a quadrature formula in the case $h_1 = h_2 = h$ can be represented as:

$$\frac{\partial^4 f(x,y)}{\partial x^4} + \frac{\partial^4 f(x,y)}{\partial y^4} \approx A \circ (A \circ f_{0,0}) + S \circ (S \circ f_{0,0}) = \\ = \frac{1}{h^4}\left(1512 f_{0,0} - \frac{1053}{2}Q_1 + 162\,Q_2 - \frac{27}{2}K_1\right), \quad (28)$$

where Q_1, Q_2, K_1 are defined in (13)–(15).

The proof is carried out according to a scheme similar to the proof of Lemma 2.

Next we follow *Remark 6*. We consider the fourth partial derivative $f_{xxxx}^{(IV)}(x)$ as a linear operator of the function f(x) of the variable x. We approximate using the pattern seven equidistant nodes at the segment [-h, h], symmetrically located relative to the grid origin (Pic. 2b):

$$f_{xxxx}^{(IV)}(0) = \frac{1}{h^4}\begin{pmatrix} I_0 f_0 + I_1\left(f_{-1/3} + f_{1/3}\right) + \\ + I_2\left(f_{-2/3} + f_{2/3}\right) + I_3\left(f_{-1} + f_1\right)\end{pmatrix}. \quad (29)$$

Using the method of indeterminate coefficients, we find the weight coefficients I_0, I_1, I_2, I_3 in the formula (27), similarly to the reasoning given in *Lemma 2*, so that it has the maximum algebraic order of error [8, p. 40] (see *Remark 7*). Nontrivial equations will be obtained only in the following orders of even degree x:

1) $f(x) \equiv 1;\ f_{xxxx}^{(4)}(0) = 0;\ I_0 + 2I_0 + 2I_0 + 2I_0 = 0;$

$f(x) = x^2; \quad f_{xxxx}^{(IV)}(0) = 0;$

$$3) \begin{pmatrix} I_0 0^2 + I_1\left(\left(-\frac{h}{3}\right)^2 + \left(\frac{h}{3}\right)^2\right) + \\ + I_2\left(\left(-\frac{2h}{3}\right)^2 + \left(\frac{2h}{3}\right)^2\right) + \\ + I_3\left((-h)^2 + h^2\right)\end{pmatrix} = $$

$= 0;\ I_1 + 4I_2 + 9I_3 = 0;$

$f(x) = x^4; \quad f_{xxxx}^{(IV)}(0) = 24;$

$$5) \begin{pmatrix} I_0 0^4 + I_1\left(\left(-\frac{h}{3}\right)^4 + \left(\frac{h}{3}\right)^4\right) + \\ + I_2\left(\left(-\frac{2h}{3}\right)^4 + \left(\frac{2h}{3}\right)^4\right) + \\ + I_3\left((-h)^4 + h^4\right)\end{pmatrix} = $$

$= 0;\ I_1 + 16I_2 + 81I_3 = 972;$

$f(x) = x^6; \quad f_{xxxx}^{(IV)}(0) = 360x^2\big|_{x=0} = 0;$

$$7) \begin{pmatrix} I_0 0^6 + I_1\left(\left(-\frac{h}{3}\right)^6 + \left(\frac{h}{3}\right)^6\right) + \\ + I_2\left(\left(-\frac{2h}{3}\right)^6 + \left(\frac{2h}{3}\right)^6\right) + \\ + I_3\left((-h)^6 + h^6\right)\end{pmatrix} = $$

$= 0;\ I_1 + 64I_2 + 729I_3 = 0.$

We obtain SLAE and its solution:

$$\begin{Bmatrix} I_0 + 2I_1 + 2I_2 + 2I_3 = 0;\ I_1 + 4I_2 + 9I_3 = 0; \\ I_1 + 16I_2 + 81I_3 = 972;\ I_1 + 64I_2 + 729I_3 = 0\end{Bmatrix}.$$

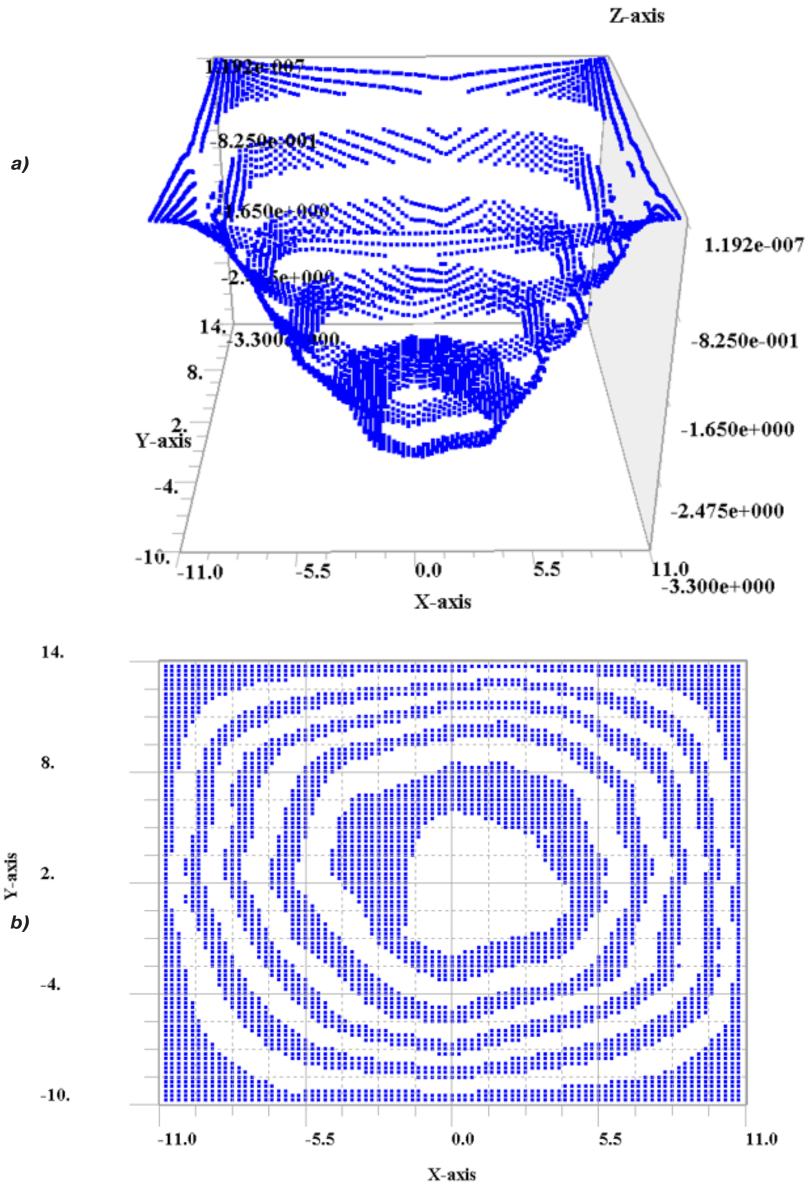
$$\left\{I_0 = 756, I_1 = -\frac{1053}{2}, I_2 = 162, I_3 = -\frac{27}{2}\right\}.$$

Then from this for (27) we obtain a quadrature formula with the found coefficients:

$$f_{xxxx}^{(IV)}(0) = \frac{1}{h^4}\begin{pmatrix} 756 f_0 - \frac{1053}{2}\left(f_{-1/3} + f_{1/3}\right) + \\ + 162\left(f_{-2/3} + f_{2/3}\right) - \frac{27}{2}\left(f_{-1} + f_1\right)\end{pmatrix} + \quad (30) \\ + O\left(h^4\right).$$

After arguments, similar to those, given in *Lemma 2* following the formula (23), it follows that the error order of the numerator of the right-hand side of (28) is proved and it is equal to eight, i.e. $O(h^8)$. Therefore, $O(h^4) = O(h^8)/h^4$.

For the function of two variables f(x, y), by analogy with the arguments given in *Lemma 2*, we obtain:



Pic. 4 a, b. Numerical solution of the Dirichlet problem for the Poisson equation: a) numerical solution of the problem (4) using the formula (13) and the right part in the form of a QR code using the formula (5) (second program), b) fields of level lines constructed according to Pic. 4a.

$$\frac{\partial^4 f(x,y)}{\partial x^4}+\frac{\partial^4 f(x,y)}{\partial y^4}=A\circ(\mathrm{A}\circ f_{0,0})+S\circ(\mathrm{S}\circ f_{0,0})=$$

$$=\frac{1}{h_1^4}\left(756f_{0,0}-\frac{1053}{2}\left(f_{-1/3,0}+f_{1/3,0}\right)+162\left(f_{-2/3,0}+f_{2/3,0}\right)-\frac{27}{2}\left(f_{-1,0}+f_{1,0}\right)\right)+$$

$$+\frac{1}{h_2^4}\begin{pmatrix}756f_{0,0}-\frac{1053}{2}\left(f_{0,-1/3}+f_{0,1/3}\right)+\\+162\left(f_{0,-2/3}+f_{2/3,0}\right)-\frac{27}{2}\left(f_{0,-1}+f_{0,1}\right)\end{pmatrix}+\qquad(31)$$

$$+O\left(h_1^4+h_2^4\right).$$

For equal steps $h_1 = h_2 = h$ the right side of the formula (29) takes the form:

$$\frac{1}{h^4}\left(1512f_{0,0}-\left(\frac{1053}{2}\right)\left(f_{\frac{1}{3},0}+f_{-\frac{1}{3},0}+f_{0,\frac{1}{3}}+f_{0,-\frac{1}{3}}\right)+\begin{matrix}162\left(f_{\frac{2}{3},0}+f_{-\frac{2}{3},0}+f_{0,\frac{2}{3}}+f_{0,-\frac{2}{3}}\right)-\\-\left(\frac{27}{2}\right)\left(f_{1,0}+f_{-1,0}+f_{0,1}+f_{0,-1}\right)\end{matrix}\right).$$

The formula (28) is proved.

The proof of Lemma 3 is completed.

Lemma 4. The fourth mixed derivative $\left.\frac{\partial^4 f(x,y)}{\partial x^2\partial y^2}\right|_{\substack{x=x_m\\y=y_n}}$ with the accuracy $O(h^4)$ in the case $h_1 = h_2 = h$ is represented in the form:

Table 1

x	y	numerical	exact
0.000000000000000E+000	0.000000000000000E+000	0.000000000000000E+000	0.000000000000000E+000
1.25663706143592	0.000000000000000E+000	0.951056516295154	0.951056516295154
2.51327412287183	0.000000000000000E+000	0.587785252292473	0.587785252292473
0.000000000000000E+000	1.25663706143592	0.951056516295154	0.951056516295154
1.25663706143592	1.25663706143592	0.242612919304468	0.242612919291777
2.51327412287183	1.25663706143592	0.464240247098885	0.464240247090147
0.000000000000000E+000	2.51327412287183	0.587785252292473	0.587785252292473
1.25663706143592	2.51327412287183	0.415266199954766	0.415266199945460
2.51327412287183	2.51327412287183	0.450894988539345	0.450894988533041

$$\left.\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2}\right|_{\substack{x=x_m\\ y=y_n}} = \frac{1}{h^4}\begin{pmatrix} -100 f_{0,0} - \frac{160}{3}K_2 + (\frac{16}{3})^2 Z_1 + \\ + \frac{10}{3}K_1 - \frac{16}{9}Z_2 + \frac{1}{9}Z_3 \end{pmatrix} + O(\mathrm{h}^4), \quad (32)$$

where K_1, K_2, Z_1, Z_2, Z_3 are defined in (15).

We carry out the proof constructively on the basis of the formula (23). According to the second variable of the function f(x, y), for each term of the formula (23), we apply a second-order finite difference operator defined in the proof of *Lemma 2*. To explain this, we use the fourth grid pattern (see Pic. 2b). We also use the symmetry of applying the differentiation operator in (21). In this case, the first argument with respect to the variable x cannot change, the second indices of the node values change according to the formula (23). Since the error in the variable x is of order $O(h^4)$, then the initial number of terms q will increase by no more than q^2 times, that is, the number of terms q^2 will be finite and will have an error of the form $O(h^4)$ with the classical properties of the «O large». We build the formula for the case $h_1 = h_2 = h$ (33).

That is, we get (30).

The proof of Lemma 4 is complete.

Next, we substitute all proved formulas (19), (22), (30) into the blank (18) and we obtain (13). That is, ***Theorem 1*** is proved.

4. Testing the formula (13)

Using a simple iteration formula (13) and a test example (8) with a solution written through elementary functions (9), we compose a program using modern high-level language Fortran [10] that supports maximum solution arrays. Let us calculate the residuals between the difference numerical solution of the problem (8) and the projection of the exact solution (9) onto the grid nodes of the base grid (6) according to the Chebyshev norm. For example, the program with a «coarse» given number of iterations and parameters of values specifying the number of points in the area m = 2000, $n_1 = n_2 = 10$, calculates the Chebyshev norm (module of the maximum difference between the numerical and exact value at the grid node) for the residual $2.113^{(e-7)}$. And for a given number of iterations and parameters of values that specify the number of points in the area m = 2000, $n_1 = n_2 = 20$, the

$$f^{(\mathrm{IV})}_{xxyy}(x,y)_{\substack{x=x_m\\ y=y_n}} = \frac{1}{h^2}\left(-10(S\circ(f_{0,0})) + \frac{16}{3}\left(S\circ(f_{-1/2,0}) + S\circ(f_{1/2,0})\right) - \frac{1}{3}\left(S\circ(f_{-1,0}) + S\circ(f_{1,0})\right)\right) =$$
$$= \frac{1}{h^4}\left(-10\left(-10 f_{0,0} + \frac{16}{3}\left(f_{0,-1/2} + f_{0,1/2}\right) - \frac{1}{3}\left(f_{0,-1} + f_{0,1}\right)\right) + \frac{16}{3}\left(-10 f_{-1/2,0} + \frac{16}{3}\left(f_{-1/2,-1/2} + f_{-1/2,1/2}\right) - \frac{1}{3}\left(f_{-1/2,-1} + f_{-1/2,1}\right)\right) + \right.$$
$$+ \frac{16}{3}\left(-10 f_{1/2,0} + \frac{16}{3}\left(f_{1/2,-1/2} + f_{1/2,1/2}\right) - \frac{1}{3}\left(f_{1/2,-1} + f_{1/2,1}\right)\right) - \frac{1}{3}\left(-10 f_{-1,0} + \frac{16}{3}\left(f_{-1,-1/2} + f_{-1,1/2}\right) - \frac{1}{3}\left(f_{-1,-1} + f_{-1,1}\right)\right) - \quad (33)$$
$$\left. - \frac{1}{3}\left(-10 f_{1,0} + \frac{16}{3}\left(f_{1,-1/2} + f_{1,1/2}\right) - \frac{1}{3}\left(f_{1,-1} + f_{1,1}\right)\right)\right) = \frac{1}{h^4}\left(100 f_{0,0} - \frac{160}{3}\left(f_{0,-1/2} + f_{0,1/2} + f_{-1/2,0} + f_{1/2,0}\right) + \right.$$
$$+ \frac{10}{3}\left(f_{0,-1} + f_{0,1} + f_{-1,0} + f_{1,0}\right) + \left(\frac{16}{3}\right)^2\left(f_{-1/2,-1/2} + f_{-1/2,1/2} + f_{1/2,-1/2} + f_{1/2,1/2}\right) - \left(\frac{16}{9}\right)\left(f_{-1/2,-1} + f_{-1/2,1} + f_{1/2,-1} + f_{1/2,1} + \right.$$
$$\left.\left. + f_{-1,-1/2} + f_{-1,1/2} + f_{1,-1/2} + f_{1,1/2}\right) + \left(\frac{1}{9}\right)\left(f_{-1,-1} + f_{-1,1} + f_{1,-1} + f_{1,1}\right)\right) + O\left(h^4\right).$$

program returns the Chebyshev norm for the residual $3.306e^{(-9)}$, that is, the order of the error of the obtained algorithm is approximately $2.113e^{(-7)}/3.306e^{(-9)} = 61.91 = 2^6$. That is, the order of the error is 6 $O(h^6)$! (In parallel, several independent programs are created (see *Remark 1*). Thus, the testing of the algorithm (10)–(31) according to the analytical example (8), (9) was carried out by a separate (first) program not related to the QR code. But the tested core of the first program was used in the second program for the QR code and using the formula (5)).

Let us give in more detail an example of operation of the first program for test example (8) using the formula (13) with a solution written through elementary functions (9) with m = 10000 (the number of iterations), and $n_1 = n_2 =$ 50 (the number of intervals of a uniform grid along the axes x, y):

h_1 = 6.283185307179587E-002,

h_2 = 6.283185307179587E-002.

A brief excerpt from the table of numerical solutions and comparisons with the exact solution is shown in Table 1 at p. 38.

The maximum value of the Chebyshev norm is Norma C = 1,356223466864038E-11. From Table 1 it is clear that the difference in solutions occurs only from 11th decimals.

In this calculation, the constants take the following values: $m = 10^4$, $n_1 = 88$, $n_2 = 96$, $m_1 =$ 22, $m_2 = 24$, $x_{min} = -11$, $x_{max} = 11$, $y_{min} = -10$, $y_{max} = 14$.

Then, the well-known «watermark» technology described in the literature quoted in [3] is used, and a suitable, agreed with «A», container is selected. A container with information is transmitted to «A», which has a «RO» recovery program prepared in advance. It is possible to transmit several different projections of the solution in order to restore the original with a lower probability of error.

Some analogy regarding the applied method can be traced in [11].

Conclusions. A new specific version of application of various mathematical methods to build a system of mathematical substantiation of the possibility of transferring QR codes with the help of steganography tools with a high degree of reliability is proposed. The possibility of building software in the form of specialized applications for the use in the transport industry, for example, to transmit information on movement of cargo shipments, is shown. The reliability of results is confirmed by the rigorous mathematical constructions, tested with the account for the results of previously published works, as well as of their longtime application in various fields, such as various types of calculations in medicine, plasma physics, in theory and practice of pattern recognition and methods of the inverse scattering problem, electrostatics and magnetostatics, hydrodynamics, etc. The advantage of the method is its ability to apply currently available developments and programs in the specified application areas.

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