



OPTIMAL MAINTENANCE OF DISCRETE OBJECTS ON THE PLANE, IN SPACE AND IN GIVEN BOUNDARIES OF THE SPHERE

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ABSTRACT

The authors of the article simulate a mathematical solution of problems associated with optimal distribution and maintenance of discrete objects belonging, particularly, to transport environment. Options for location from one to ten objects, which are intended to be in space, on a straight line or on the

plane, within the boundaries of a sphere of a certain radius, are considered. The results of the study can be applied for planning residential areas, optimal location of public transport stop points on roads and of transport rescue services. Following the suggested model, similar calculations can be used for any aspect of human life.

Keywords: transport, mathematical model, distribution law, Weber problem, Fermat problem, discrete objects, optimal location, plane, space, line, sphere.

Background. Optimal allocation of transport objects and facilities is one of the problems having key impact on minimization of costs of transport services. We have studied some models of mathematical solution of this problem with regard to different conditions. The task itself assumes that there are n arranged objects A_1, A_2, \dots, A_n . It is required to find placements for p -points of service C_1, C_2, \dots, C_p in such a way as to minimize the total costs of the upcoming servicing. It is understood that each object is tied to one point, and each point can potentially serve any number of clients.

There are many statements of similar problems, comprising cases when the choice of location of such points is limited to a set of points specified in advance, for example, by vertices of a graph. Such a problem posed on a graph of a general form (on finding the so-called p -median of a graph¹), as it is known, belongs to the class of NP-complete [nondeterministic polynomial, 1, pp. 193–195]. Each point serves a whole cluster of objects, and complexity of the solution is associated with a rapid increase in the number of options of splitting into clusters. With introduction of restrictions on these options, it is possible to develop efficient algorithms. For example, in [2, pp. 60–62], $O(pn^2)$ algorithm was proposed in the case where the graph is a tree. In [3] the case is considered when location of points is limited to a certain line in three-dimensional space. In [4–7] problems of optimal location of a highway, optimal distribution of funds between programs for traffic safety in transport, optimal location of social facilities with normal and uniform distribution are studied.

Here we assume that each cluster served by this point is a «continuous» segment of objects, i.e. if C_1 serves A_1 and A_2 , then it serves all the objects in between – A_2, A_3, A_4 . Thus, it is required to optimally split the set of served facilities or objects (to which services are provided) into p -«continuous» segments and for each segment to find the optimal location of the required service points.

Objective. The objective of the authors is to consider different variants of optimal location of discrete served and servicing objects on the plane, in space and in given boundaries of the sphere.

Methods. The authors use dynamic programming, distribution law, geometric and algebraic methods (Weber problem, Fermat problem solution, curvilinear integral method).

¹ The problem of finding the p -median of a given graph is the problem of locating a given number of service p -points, when the sum of the shortest distances from the vertices of the graph to the nearest points is minimal.

Results.

Mathematical formulation of the problem

Let i_1, i_2, \dots, i_p – be indices for splitting the list of facilities served into zones, i.e. A_1, \dots, A_{i_1} – first service zone, which is served by the service point C_1 , the zone $A_{i_1+1}, \dots, A_{i_2}$ – by the point C_2 , etc. Let d_{ij} – be distance from the object A_i to the point C_j and m_1, \dots, m_n – be «weights» of facilities served. Then the costs associated with the point C_j are:

$$F_j = \sum_{i=i_{j-1}+1}^{i_j} m_i d_{ij}.$$

Total maintenance costs:

$$F = \sum_{j=1}^p F_j.$$

It is required to find the minimum F in all possible sets i_1, i_2, \dots, i_p and the locations of the served facilities A_1, \dots, A_n ($i \leq j$).

Algorithm of the solution

We use a dynamic programming method as a core one.

Let's denote by $G_k(i, j)$ – the minimum total costs for the subtask, in which only objects A_1, \dots, A_j ($i \leq j$, $k \leq j - i + 1$) participate, and the required number of points is equal to k .

1. Using the gradient descent method for all zones A_1, \dots, A_n ($i \leq j$) we find the optimal location and optimal costs $G_1(i, j)$ in the case of a single service point. This is the well-known Weber's task.

2. We find the optimal location of two points and optimal costs:

$$G_2(i, n) = \min_{i \leq k < n} (G_1(i, k) + G_1(k+1, n)), i = 1, 2, \dots, n-1.$$

3. And so on

$$G_3(i, n) = \min_{i \leq k < n-1} (G_1(i, k) + G_2(k+1, n)), i = 1, 2, \dots, n-2;$$

$$G_p(1, n) = \min_{i \leq k < n-p+2} (G_1(i, k) + G_{p-1}(k+1, n)).$$

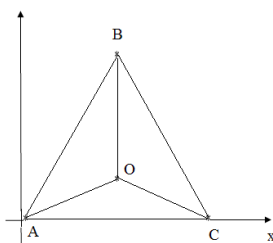
Solving tasks on the plane

Let's consider a number of examples on the plane. Let the coordinates of the served facilities be $A(x, y)$, and the coordinates of the service point be $C_j(x, y)$, then

$$d_{ij} = \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

Example 1. There are three objects $A(0; 0)$, $B(5; 10)$, $C(10; 0)$. It is required to optimally place the rescue helicopter service if all three objects are equally important.

This is the well-known Fermat's problem: to find such a point O (see Pic. 1), that the sum of the distances $OA + OB + OC$ is minimal. If the angles of



Pic. 1. Fermat point of the triangle ABC.

the triangle are less than 120° (in our case, they are sharp), then the desired point is the only one (Fermat point) and the angles $\angle AOC$, $\angle AOB$, $\angle BOC$ are equal to 120° . Calculations with the algorithm give $O(5; 2,85)$.

Example 2. In the previous problem we change the coordinates to $A(0; 0)$, $B(-5; 5)$, $C(10; 0)$. In this case the angle $\angle BAC = 135^\circ$ is obtuse and a minimum is reached at point A. Calculations according to the algorithm give $O(0; 0)$.

Example 3. In the conditions of example 1, let point B be one and a half times more important than points A and C. In this case, it is necessary to find point O so that the sum of the distances $OA + 1,5OB + OC$ is minimal. Calculations according to the algorithm give $O(5; 5,655)$.

Example 4. In the conditions of example 1 let's find the optimal location of two rescue helicopter services.

Here $F_{\min} = 10$: O_1 at any point on the side AC, and O_2 at point B.

Example 5. There are four objects $A(0; 0)$, $B(20; 0)$, $C(20; 20)$, $D(0; 20)$ (forming the route ABCD). It is necessary to optimally place:

- one rescue helicopter service, if all objects are equally important;
- two rescue helicopter services, if all objects are equally important.

Solution:

a) In this case, the answer is obvious: the point is at the intersection of the diagonals of the square. The result of the program – the coordinates of the rescue service $(10; 10)$;

b) The result of the software application:

$F_{\min} = 38,63708$.

opt. option No. 1

$x(1) = 0$; $y(1) = 0$.

$x(2) = 15,75$; $y(2) = 15,75$.

opt. option No. 2

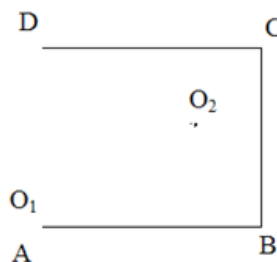
$x(1) = 15,75$; $y(1) = 4,25$.

$x(2) = 0$; $y(2) = 20$.

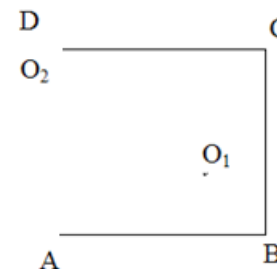
Thus, there are two optimal symmetric options.

The first: $O_1(0, 0)$, and $O_2(15,75; 15,75)$ is the Fermat point for the triangle BCD (see Pic. 2).

The second: O_2 at point D, and $O_1(15,75; 4,25)$ – the Fermat point for the triangle ABC (see Pic. 3).



Pic. 2. The first option of optimal location of two rescue services.



Pic. 3. The second option of optimal location of two rescue services.

Note. Two more obvious options for the square: O_1 at point B, and O_2 – the Fermat point for the triangle ADC;

O_1 at point C, and O_2 – the Fermat point for the triangle ADB.

These options are not suitable for the considered model, since we are not dealing with a square, but with a route ABCD (there is no connection of A with D). Cyclic routes will be considered later.

Example 6. There are seven settlements $A_1(0,0)$, $A_2(2,1)$, $A_3(4,3)$, $A_4(6,4)$, $A_5(8,3)$, $A_6(10,1)$, $A_7(12,0)$ (see Pic. 4). It is necessary to place optimally:

- one rescue helicopter service, if all objects are equally important;
- two rescue helicopter services, if all objects are equally important;
- three rescue helicopter services, if all objects are equally important;
- three rescue helicopter services if the population figures are $1 : 1,3 : 1,5 : 2 : 2,5 : 3 : 3,5$.

Solution:

a) The result of the program application:

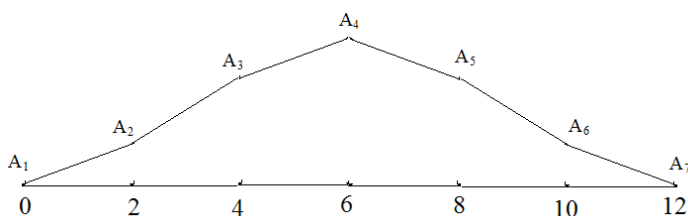
$F_{\min} = 27,16705$.

opt. option only one

$x(1) = 6$; $y(1) = 2,51$.

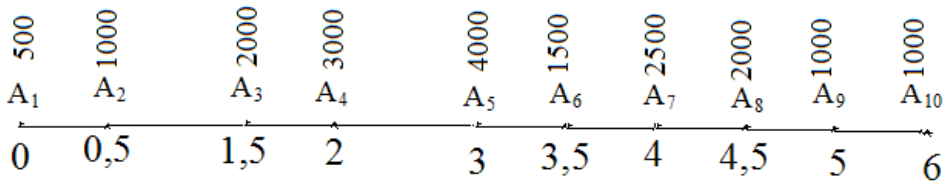
There is one optimal option for location of the rescue service $O(6; 2,51)$.

b) The result of the program:



Pic. 4. Location of seven settlements.





Pic. 5. Location of ten settlements on the straight line.

$$F_{\min} = 15,10434.$$

opt. option No. 1

$$x(1) = 2; \quad y(1) = 1.$$

$$x(2) = 8,85; \quad y(2) = 2,13.$$

opt. option No. 2

$$x(1) = 3,15; \quad y(1) = 2,13.$$

$$x(2) = 10; \quad y(2) = 1.$$

There are two optimal symmetric options.

The first: one point $O_1(2, 1)$ – the Fermat point for the triangle $A_1A_2A_3$; and $O_2(8,85; 2,13)$ – the intersection point of the diagonals of the quadrilateral $A_4A_5A_6A_7$.

The second: $O_3(10, 1)$ – the Fermat point for the triangle $A_4A_5A_6$; and $O_4(3,15; 2,13)$ – the intersection point of the diagonals of the quadrilateral $A_1A_2A_3A_4$.

c) The result of the program:

$$F_{\min} = 8,944272.$$

opt. option only one

$$x(1) = 1,8; \quad y(1) = 0,9.$$

$$x(2) = 6; \quad y(2) = 4.$$

$$x(3) = 10,2; \quad y(3) = 0,9.$$

There is one optimal way to place the rescue service $O_1(1,8; 0,9)$ – a point on the section A_1A_2 (which means that it is possible to take any point of this segment instead); $O_2(6; 4)$ – the Fermat point for the triangle $A_3A_4A_5$; $O_3(10,2; 0,9)$ – a point on the section A_6A_7 (which means that it is possible to take any point of this section instead).

d) The result of the program application:

$$F_{\min} = 17,6591.$$

opt. option only one

$$x(1) = 2; \quad y(1) = 1.$$

$$x(2) = 8; \quad y(2) = 3.$$

$$x(3) = 12; \quad y(3) = 0.$$

There is one optimal location option: $O_1(2; 1)$ – the Fermat point for the triangle $A_1A_2A_3$; $O_2(8; 3)$ – the Fermat point for the triangle $A_4A_5A_6$ and $O_3(12; 0)$ are located at the point A_7 . Rescue services have shifted to the right.

Solving problems on the straight line

The method of optimal location of rescue services on the plane is applicable for optimal location on the line.

Example 7. There are 10 settlements located on the line:

$A_1(0)$; $A_2(0,5 \text{ km})$; $A_3(1,5 \text{ km})$; $A_4(2 \text{ km})$; $A_5(3 \text{ km})$; $A_6(3,5 \text{ km})$; $A_7(4 \text{ km})$; $A_8(4,5 \text{ km})$; $A_9(5 \text{ km})$; $A_{10}(6 \text{ km})$ (see Pic. 5). In these settlements 500, 1000, 2000, 3000, 4000, 1500, 2500, 2000, 1000, 1000 people, respectively, live.

It is necessary to optimally place: a) one; b) two; c) three; d) four; e) five bus stops.

Solution:

a) We get

$$F_{\min} = 42,5.$$

opt. option only one

$$x(1) = 3; \quad y(1) = 0.$$

One bus stop shall be placed at point A_5 .

b) Optimal location of two bus stops: at A_3 and A_7 .

$$F_{\min} = 24.$$

opt. option only one

$$x(1) = 1,5; \quad y(1) = 0.$$

$$x(2) = 4; \quad y(2) = 0.$$

c) Optimal location of three bus stops: at A_3 , A_5 and A_8 .

$$F_{\min} = 14,5.$$

opt. option only one

$$x(1) = 1,5; \quad y(1) = 0.$$

$$x(2) = 3; \quad y(2) = 0.$$

$$x(3) = 4,5; \quad y(3) = 0.$$

d) Optimal location of four bus stops: at A_2 , A_4 , A_5 and A_8 .

$$F_{\min} = 10,5.$$

opt. option only one

$$x(1) = 0,5; \quad y(1) = 0.$$

$$x(2) = 2; \quad y(2) = 0.$$

$$x(3) = 3; \quad y(3) = 0.$$

$$x(4) = 4,5; \quad y(4) = 0.$$

e) Optimal location of five bus stops: at A_2 , A_4 , A_5 , A_8 and A_{10} .

$$F_{\min} = 7,5.$$

opt. option only one

$$x(1) = 0,5; \quad y(1) = 0.$$

$$x(2) = 2; \quad y(2) = 0.$$

$$x(3) = 3; \quad y(3) = 0.$$

$$x(4) = 4,5; \quad y(4) = 0.$$

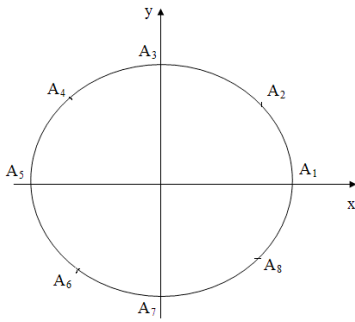
$$x(5) = 6; \quad y(5) = 0.$$

Cyclic routes

Let there be n arranged served objects A_1, A_2, \dots, A_n forming a cycle, i.e. connections between any neighboring objects and between A_1 and A_n . As for the route (the ends have no connection), it is necessary to optimally split the served facilities or objects into m inseparable zones, and to find for each zone the optimal location of the service point so that the total costs are minimal.

Considering $n - m + 1$ route problems (A_1, A_2, \dots, A_n ; $A_2, A_3, \dots, A_n, A_1$; $A_3, A_4, \dots, A_n, A_1, A_2$ etc.), any optimal location for the cycle is among the optimal locations of the routes. Therefore, as applied to the cycle, it is necessary to find optimal locations for each individual route and then to choose less costly ones among them.

Example 8. The condition of example 5: there are four major objects $A(0; 0)$, $B(20; 0)$, $C(20; 20)$, $D(0; 20)$ (forming a cycle). The task is to optimally



Pic. 6. The location of landing grounds.

place two rescue helicopter services, if all objects are equally important.

Solution: It is necessary to consider $4 - 2 + 1 = 3$ routes.

1. For the route ABCD in example 5 the answer is already given:

$$F_{\min} = 38,63708.$$

opt. option No. 1

$$x(1) = 0; \quad y(1) = 0.$$

$$x(2) = 15,75; \quad y(2) = 15,75.$$

opt. option No. 2

$$x(1) = 15,75; \quad y(1) = 4,25.$$

$$x(2) = 0; \quad y(2) = 20.$$

There are two optimal symmetric options.

The first: $O_1(0, 0)$ and $O_2(15,75; 15,75)$ – the Fermat point for the triangle BCD (see Pic. 2).

The second: O_2 at point D, and $O_1(15,75; 4,25)$ – the Fermat point for the triangle ABC (see Pic. 3).

2. For the route BCDA we have:

$$F_{\min} = 38,63708.$$

opt. option No. 1

$$x(1) = 20; \quad y(1) = 0.$$

$$x(2) = 4,25; \quad y(2) = 15,75.$$

opt. option No. 2

$$x(1) = 15,75; \quad y(1) = 15,75.$$

$$x(2) = 0; \quad y(2) = 0.$$

There are two optimal symmetric options.

The first: $O_1(20, 0)$, and $O_2(4,25; 15,75)$ – the Fermat point for the triangle ACD.

The second: O_2 at point A, and $O_1(15,75; 15,75)$ – the Fermat point for the triangle BCD (this option already exists).

3. For the route CDAB we have:

$$F_{\min} = 38,63708.$$

opt. option No. 1

$$x(1) = 20; \quad y(1) = 20.$$

$$x(2) = 4,25; \quad y(2) = 4,25.$$

opt. option No. 2

$$x(1) = 4,25; \quad y(1) = 15,75.$$

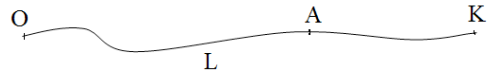
$$x(2) = 20; \quad y(2) = 0.$$

There are two optimal symmetric options.

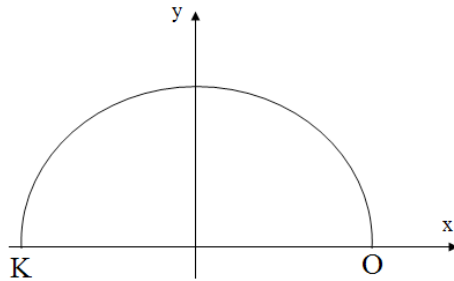
The first: $O_1(20; 20)$, and $O_2(4,25; 4,25)$ – the Fermat point for the triangle ADB.

The second: O_2 at point B, and $O_1(4,25; 15,75)$ – the Fermat point for the triangle ACD (this option already exists).

We obtain four optimal location options for the cycle ABCD:



Pic. 7. Schematic location of the road.



Pic. 8. Scheme of a section of a homogeneous railway.

1. $O_1(0; 0)$, and $O_2(15,75; 15,75)$.

2. $O_1(15,75; 4,25)$, and $O_2(0; 20)$.

3. $O_1(20; 0)$, and $O_2(4,25; 15,75)$.

4. $O_1(20; 20)$, and $O_2(4,25; 4,25)$.

Example 9. For providing rescue services for a circle route, representing a circle with a radius of 25 km, 8 landing grounds A_1, A_2, \dots, A_8 (see Pic. 6), are located at the same distance from each other. The intensity of passenger flows is correlated as $1,5 : 1,8 : 2 : 1,7 : 1,6 : 1,4 : 1 : 1,3$ respectively. It is required to analyze and find the optimal location for:

- one helicopter rescue service;
- two helicopter rescue services;
- three helicopter rescue services.

Solution:

a) There is only one optimal location option for one rescue service.

$$F_{\min} = 302,6217.$$

$$x(1) = -0,247; \quad y(1) = 6,13.$$

$O(-0,247 \text{ km}; 6,13 \text{ km})$ is optimum.

b) From among the routes $A_1, A_2, \dots, A_8; A_2, A_3, \dots, A_1; A_3, A_4, \dots, A_2; A_4, A_5, \dots, A_3; A_5, A_6, \dots, A_4; A_6, A_7, \dots, A_5; A_7, A_8, \dots, A_6$ minimum costs with optimal location of two helicopter rescue services are associated with the route A_4, A_5, \dots, A_3 .

$$F_{\min} = 206,4624.$$

$$x(1) = -22,74; \quad y(1) = -0,64.$$

$$x(2) = 16,2; \quad y(2) = 15,11.$$

$O_1(-22,74 \text{ km}; -0,64 \text{ km}); O_2(16,2 \text{ km}; 15,11 \text{ km})$ – optimal location of two helicopter rescue services.

c) From the analysis of the results for all routes two optimal location options were received:

$$F_{\min} = 206,4624.$$

opt. option No. 1

$$x(1) = 0; \quad y(1) = 25.$$

$$x(2) = -25; \quad y(2) = 0.$$

$$x(3) = 17,68; \quad y(3) = 17,68.$$

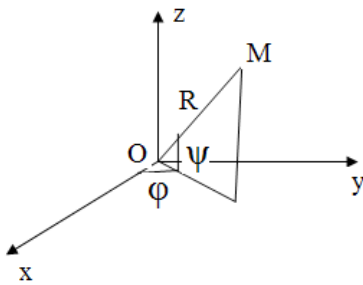
opt. option No. 2

$$x(1) = 0; \quad y(1) = 25.$$

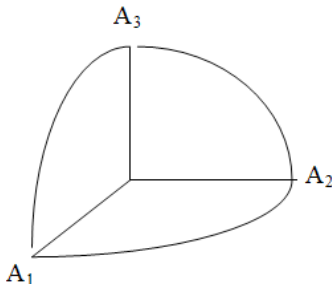
$$x(2) = -17,68; \quad y(2) = -17,68.$$

$$x(3) = 25; \quad y(3) = 0.$$





Pic. 9. Sphere of radius R.



Pic. 10. The location of the route $A_1A_2A_3$ on the sphere.

So, there are two optimal accommodation options for three rescue services.

The first: $O_1(0; 25 \text{ km}); O_2(-25 \text{ km}; 0); O_3(17,68 \text{ km}; -17,68 \text{ km})$.

The second: $O_1(0; 25 \text{ km}); O_2(-17,68 \text{ km}; -17,68 \text{ km}); O_3(25 \text{ km}; 0 \text{ km})$.

Continuous location of objects

Let's suppose that there is a curve L (on a plane or in space) on which served objects are placed (for example, a railway or road). Let A be a point on the curve, O be the starting point, and K be the end point of the curve (see Pic. 7). Let s be the length of the curve from O to A. To service an object at point A(s), there is a weighting coefficient $m(s)$. Let C be some point from which maintenance will be provided. Then maintenance costs are equal to the curvilinear integral

$$\int_0^K |CA| \cdot m(s) ds.$$

The task is to find the optimal location of the point C, corresponding to the minimum costs.

Let's call the more general task: to break the curve L into m parts, and to find for each part the optimal location of the service point so that the total maintenance costs are minimal.

It is not analytically possible to solve the task (with the exception of trivial tasks), but it is possible to make approximations using the above method. Since the curvilinear integral is the limit of the integral sum (the partitioning step is made constant), and for the integral sum we can find the optimal location, then with a large number of partitions we get the desired approximate solution.

Let's consider specific examples:

Example 10. For a section of a homogeneous ($m(s) = 1$) railway, which is a semicircle with a radius of 30 km (see Pic. 8), it is necessary to find the optimal location for:

- one rescue helicopter service;
- two rescue helicopter services;
- three rescue helicopter services.

Solution:

a) Results of optimal location options with different number of partitions:

partitions = 20	partitions = 40
$F_{\min} = 444,6107.$	$F_{\min} = 871,9875.$
opt. option	opt. option
$x(1) = 0; y(1) = 22,85.$	$x(1) = 0; y(1) = 23,26.$
partitions = 80	partitions = 100
$F_{\min} = 1726,943.$	$F_{\min} = 2154,438.$
opt. option	opt. option
$x(1) = 0; y(1) = 23,43.$	$x(1) = 0; y(1) = 23,48.$

The optimal location (with an accuracy of 50 meters) is located at point O(0; 23,48 km).

b) Results of optimal location in case of different number of partitions:

partitions = 20	partitions = 40
$F_{\min} = 241,5657.$	$F_{\min} = 471,9252.$
opt. option	opt. option
$x(1) = 20,90; y(1) = 19,33.$	$x(1) = 20,64; y(1) = 19,88.$
$x(2) = -20,90; y(2) = 19,33.$	$x(2) = -20,63; y(2) = 19,89.$
partitions = 80	partitions = 100
$F_{\min} = 932,9471.$	$F_{\min} = 1163,457.$
opt. option	opt. option
$x(1) = 20,47; y(1) = 20,14.$	$x(1) = 20,44; y(1) = 20,17.$
$x(2) = -20,47; y(2) = 20,13.$	$x(2) = -20,44; y(2) = 20,18.$

O(0; 23,48 km).

b) Results of optimal location options with different number of partitions:

The optimal location (with an accuracy of 40 meters) is located at points $O_1(20,44 \text{ km}; 20,17 \text{ km}); O_2(-20,44 \text{ km}; 20,18 \text{ km})$.

c) Results of optimal location options with different number of partitions:

partitions = 80	partitions = 100
$F_{\min} = 630,09.$	$F_{\min} = 785,7234.$
opt. option	opt. option
$x(1) = 25,63; y(1) = 14,60.$	$x(1) = 25,78; y(1) = 14,37.$
$x(2) = 0; y(2) = 29,50.$	$x(2) = 0; y(2) = 29,46.$
$x(3) = -25,64; y(3) = 14,59.$	$x(3) = -25,78; y(3) = 14,37.$

The optimal location (with an accuracy of 200 meters) is located at points $O_1(25,75 \text{ km}; 14,37 \text{ km}); O_2(0; 29,46 \text{ km}); O_3(-25,78 \text{ km}; 14,37 \text{ km})$.

Example 11. The conditions of the previous example are maintained, but the road is heterogeneous – $m(\phi) = 1 + \phi/\pi$, i.e. from O($m(0) = 1$) to K($m(\pi) = 2$) road conditions worsen. It is required to find the optimal location for:

- one rescue helicopter service;
- two rescue helicopter services.

Solution:

a) Results of optimal location options with different number of partitions:

partitions = 80	partitions = 100
$F_{\min} = 2524,94.$	$F_{\min} = 3150,322.$
opt. option	opt. option
$x(1) = -6,98; y(1) = 23,04.$	$x(1) = -6,97; y(1) = 23,08.$

The optimal location (with an accuracy of 40 meters) is located at point O(-6,97 km; 23,08 km). Due to the deterioration of the road towards K, the optimum point also shifts to the left.

b) Results of optimal placements with different number of partitions:

partitions = 80	partitions = 100
$F_{\min} = 1378,391$.	$F_{\min} = 1719,109$.
opt. option	opt. option
$x(1) = 16,44$; $y(1) = 23,22$.	$x(1) = 16,42$; $y(1) = 23,25$.
$x(2) = -23,09$; $y(2) = 17,55$.	$x(2) = -23,06$; $y(2) = 17,61$.

The optimal location (with an accuracy of 60 meters) is located at points $O_1(16,43 \text{ km}; 23,25 \text{ km})$; $O_2(-23,06 \text{ km}; 17,61 \text{ km})$.

Optimization of placement on the sphere

Let M be a point on a sphere of radius R having a longitude coordinate ϕ ($-\pi \leq \phi \leq \pi$) and a coordinate of latitude ψ ($-\pi/2 \leq \psi \leq \pi/2$) (see Pic. 9), then the vector OM has Cartesian coordinates ($R \cdot \cos \psi \cdot \cos \phi$; $R \cdot \cos \psi \cdot \sin \phi$; $R \cdot \sin \psi$).

Let us have two points on the sphere $M_1(\phi_1; \psi_1)$, $M_2(\phi_2; \psi_2)$. We denote by ω the angle between the vectors OM_1 and OM_2 , then $\cos \omega = \cos \psi_1 \cdot \cos \phi_1 \cdot \cos \psi_2 \cdot \cos \phi_2 + \cos \psi_1 \cdot \sin \phi_1 \cdot \cos \psi_2 \cdot \sin \phi_2 + \sin \psi_1 \cdot \sin \psi_2 = \cos \psi_1 \cdot \cos \psi_2 \cdot \cos(\phi_2 - \phi_1) + \sin \psi_1 \cdot \sin \psi_2$.

From here we get the formula for the distance between the points $M_1(\phi_1; \psi_1)$, $M_2(\phi_2; \psi_2)$ on the sphere:

$$D = R \cdot \arccos(\cos \psi_1 \cdot \cos \phi_1 \cdot \cos \psi_2 \cdot \cos \phi_2 + \cos \psi_1 \cdot \sin \phi_1 \cdot \cos \psi_2 \cdot \sin \phi_2 + \sin \psi_1 \cdot \sin \psi_2) = R \cdot \arccos(\cos \psi_1 \cdot \cos \psi_2 \cdot \cos(\phi_2 - \phi_1) + \sin \psi_1 \cdot \sin \psi_2).$$

Now, similarly to the previous chapters, we solve the problem of the optimal location of service points for objects located on a sphere. The algorithm for solving the problem is the same as shown previously.

Example 12. On a sphere of the radius of 6400 km, the route $A_1 A_2 A_3$ is given; where $A_1(0; 0)$; $A_2(\pi/2; 0)$; $A_3(0; \pi/2)$ (see Pic. 10). The weight coefficients of the points are the same.

It is required to find:

a) a point on the sphere for which the sum of the distances to the given points is minimal (the Fermat point on the sphere);

b) two points for which the sum of the distances from the given point to the nearest of these two points is minimal.

Solution:

a) there is only one optimal location option.

$$F_{\min} = 28,6595.$$

opt. option

$$x(1) = 0,78544 \quad y(1) = 0,6157.$$

$O(0,7854; 0,6157)$ – the center point of the eighth part of the sphere.

b) there are two optimal location options.

$$F_{\min} = 15,7114.$$

opt. option No. 1

$$x(1) = 0; \quad y(1) = 0.$$

$$x(2) = 1,57; \quad y(2) = 1,28.$$

opt. option No. 2

$$x(1) = 0,90; \quad y(1) = 0.$$

$$x(2) = 1; \quad y(2) = 1,57.$$

The first: $O_1(0; 0)$ (at point A_1), and O_2 is at any point of the arc $A_2 A_3$.

The second: O_1 – at any point of the arc $A_1 A_2$; O_2 is at point A_3 .

Note. The route $A_1 A_2 A_3$ is considered (there is no $A_1 A_3$ connection), therefore the solution option O_1 at point A_2 and O_2 at any point of the arc $A_1 A_3$ is not taken into account.

Example 13. We take the route $A_1 A_2 A_3 A_4 A_5 A_6 A_7$, where $A_1(0; 0)$; $A_2(\pi/12; \pi/12)$; $A_3(2\pi/12; 2\pi/12)$; $A_4(3\pi/12; 3\pi/12)$; $A_5(4\pi/12; 2\pi/12)$; $A_6(5\pi/12; \pi/12)$; $A_7(6\pi/12; 0)$. The weighting coefficients are related as 2:1,5:1,3:1:1,3:1,5:2. It is required to find: a) one optimal point; b) two optimal points; c) three optimal points.

Solution:

a) there is only one optimal location option.

$$F_{\min} = 60,54433.$$

opt. option

$$x(1) = 0,7857; \quad y(1) = 0,4228.$$

b) there is only one optimal location option.

$$F_{\min} = 30,8262.$$

opt. option

$$x(1) = 0,26; \quad y(1) = 0,26.$$

$$x(2) = 1,31; \quad y(2) = 0,26.$$

c) there is only one optimal location option.

$$F_{\min} = 18,93.$$

opt. option

$$x(1) = 0; \quad y(1) = 0.$$

$$x(2) = 0,785; \quad y(2) = 0,635.$$

$$x(3) = 1,57; \quad y(3) = 0.$$

Example 14. Fifty stations from St. Petersburg to Moscow with their geographical coordinates (e.g., Tver, $56^{\circ}51'28''N$, $35^{\circ}55'18''E$) are taken. The weight coefficients of the stations are equal to the number of residents of settlements. It is necessary to determine the optimal location of rescue helicopter services (or. e.g., of logistics facilities) in: a) one; b) three; c) five; d) seven; e) ten settlements.

Solution:

a) there is only one optimal location option.

$$F_{\min} = 5357,119.$$

opt. option

$$x(1) = 0,6570193; \quad 37 \quad 38 \quad 39.$$

$$y(1) = 0,9735325; \quad 55 \quad 46 \quad 45.$$

$O(55^{\circ}46'45''N$, $37^{\circ}38'39''E)$ – Moscow is the optimal location for one rescue helicopter service.

b) there is only one optimal location option.

$$F_{\min} = 131,7336.$$

opt. option

$$x(1) = 0,529522; \quad 30 \quad 20 \quad 21.$$

$$y(1) = 1,046008; \quad 59 \quad 55 \quad 54.$$

$$x(2) = 0,626340; \quad 35 \quad 53 \quad 11.$$

$$y(2) = 0,992498; \quad 56 \quad 51 \quad 57.$$

$$x(3) = 0,657666; \quad 37 \quad 40 \quad 53.$$

$$y(3) = 0,9734444; \quad 55 \quad 46 \quad 27.$$

$O_1(59^{\circ}55'54''N$, $30^{\circ}20'21''E)$ – St. Petersburg; $O_2(56^{\circ}51'57''N$, $35^{\circ}53'11''E)$; $O_3(55^{\circ}46'27''N$, $37^{\circ}40'53''E)$ – Moscow – the optimal location of three rescue helicopter services.

c) there is only one optimal location option.

$$F_{\min} = 99,93713.$$

opt. option

$$x(1) = 0,529522; \quad 30 \quad 20 \quad 21.$$

$$y(1) = 1,046008; \quad 59 \quad 55 \quad 54.$$

$$x(2) = 0,603122; \quad 34 \quad 33 \quad 22.$$

$$y(2) = 1,005301; \quad 57 \quad 35 \quad 58.$$

$$x(3) = 0,626474; \quad 35 \quad 53 \quad 39.$$

$$y(3) = 0,992484; \quad 56 \quad 51 \quad 54.$$

$$x(4) = 0,652459; \quad 37 \quad 22 \quad 59.$$

$$y(4) = 0,975852; \quad 55 \quad 54 \quad 43.$$

$$x(5) = 0,657624; \quad 37 \quad 40 \quad 44.$$

$$y(5) = 0,973372; \quad 55 \quad 46 \quad 12.$$





d) there is only one optimal location option.

$$F_{\min} = 83,10983.$$

opt. option

$x(1) = 0,529319;$	30	19	39.
$y(1) = 1,046664;$	59	58	10.
$x(2) = 0,539970;$	30	56	16.
$y(2) = 1,039119;$	59	32	13.
$x(3) = 0,602933;$	34	32	43.
$y(3) = 1,005344;$	57	36	7.
$x(4) = 0,626291;$	35	53	1.
$y(4) = 0,992527;$	56	52	3.
$x(5) = 0,641392;$	36	44	56.
$y(5) = 0,983039;$	56	19	26.
$x(6) = 0,653102;$	37	25	11.
$y(6) = 0,975525;$	55	53	36.
$x(7) = 0,657624;$	37	40	44.
$y(7) = 0,975525;$	55	46	12.

e) there is only one optimal location option.

$$F_{\min} = 72,55429.$$

opt. option

$x(1) = 0,529319;$	30	19	39.
$y(1) = 1,046664;$	59	58	10.
$x(2) = 0,539670;$	30	55	14.
$y(2) = 1,039566;$	59	33	45.
$x(3) = 0,562062;$	32	12	13.
$y(3) = 1,027465;$	58	52	9.
$x(4) = 0,603013;$	34	33	0.
$y(4) = 1,005241;$	57	35	45.
$x(5) = 0,610758;$	34	59	37.
$y(5) = 0,995543;$	57	2	25.
$x(6) = 0,626559;$	35	53	57.
$y(6) = 0,992503;$	56	51	58.
$x(7) = 0,641078;$	36	43	51.
$y(7) = 0,982985;$	56	19	15.
$x(8) = 0,649193;$	37	11	45.
$y(8) = 0,976829;$	55	58	5.
$x(9) = 0,653784;$	37	27	32.
$y(9) = 0,975248;$	55	52	39.
$x(10) = 0,657624;$	37	40	44.
$y(10) = 0,973372;$	55	46	12.

Example 15. 50 stations from St. Petersburg to Moscow are considered. The weight coefficients of stations are equal to the number of residents of settlements. It is necessary to determine the optimal location of the stops (not considering St. Petersburg and Moscow): a) one stop; b) two stops; c) four stops; d) six stops; e) eight stops; e) ten stops.

Solution:

a) there is only one optimal location option.

$$F_{\min} = 47456,63.$$

opt. option

$$x(1) = 482,95.$$

Optimal placement of one stop – 482,95 in Tver, 419363 people.

b) there is only one optimal location option.

$$F_{\min} = 36362,02.$$

opt. option

$$x(1) = 364,2.$$

$$x(2) = 482,95.$$

Optimal location of two stops – Vyshniy Volochek, 47 732 people and Tver, 419 363 people.

c) there is only one optimal location option.

$$F_{\min} = 20287,78.$$

opt. option

$$x(1) = 364,2.$$

$$x(2) = 482,95.$$

$$x(3) = 560,6.$$

$$x(4) = 631,17.$$

Optimal location of four stops – Vyshniy Volochek, 47 732 people; Tver, 419 363 people; Klin, 79 056 people and Khimki 244 668 people.

d) there is only one optimal location option.

$$F_{\min} = 13131,44.$$

opt. option

$$x(1) = 52,53.$$

$$x(2) = 364,2.$$

$$x(3) = 436,56.$$

$$x(4) = 482,95.$$

$$x(5) = 560,6.$$

$$x(6) = 631,17.$$

Optimal location of six stops – Tosno, 37 875 people; Vyshniy Volochek, 47 732 people; Likhoslavl, 46 031 people; Tver, 419 363 people; Klin, 79 056 people and Khimki, 244 668 people.

e) there is only one optimal location option.

$$F_{\min} = 8741,181.$$

opt. option

$$x(1) = 52,53.$$

$$x(2) = 161,71.$$

$$x(3) = 364,2.$$

$$x(4) = 436,56.$$

$$x(5) = 482,95.$$

$$x(6) = 560,6.$$

$$x(7) = 611,18.$$

$$x(8) = 631,17.$$

Optimal location of eight stops – Tosno, 37 875 people; Malaya Vishera, 11 015 people; Vyshniy Volochek, 47 732 people; Likhoslavl, 46 031 people; Tver, 419 363 people; Klin, 79 056 people; Kryukovo, 95 645 people; Khimki, 244 668 people.

e) there is only one optimal location option.

$$F_{\min} = 5963,371.$$

opt. option

$$x(1) = 52,53.$$

$$x(2) = 117,82.$$

$$x(3) = 319,43.$$

$$x(4) = 364,2.$$

$$x(5) = 436,56.$$

$$x(6) = 482,95.$$

$$x(7) = 560,6.$$

$$x(8) = 584,92.$$

$$x(9) = 611,18.$$

$$x(10) = 631,17.$$

Optimal location of ten stops – Tosno, 37875 people; Chudovo-Moskovskoe, 14 730 people; Bologoe-Moskovskoe, 21 425 people; Vyshniy Volochek, 47732 people; Likhoslavl, 46031 people; Tver, 419 363 people; Klin, 79 056 people; Podsolnechnaya, 52642 people; Kryukovo, 95645 people; Khimki, 244668 people.

Conclusion.

A mathematical model of optimal servicing of discrete objects, which can be located on a straight line, a plane, in space or on a sphere, is constructed. The results of the study can be applied when planning residential areas, for optimal location of transport stops on roads, optimal location of rescue services for transport, optimal distribution of other transport specific objects, etc. Similar calculations are possible for any aspect of human life.

REFERENCES

1. Megiddo, N., Supowit, K. J. On the complexity of some common geometric location problems. *SIAM Journal on Computing*, 1984, Vol. 13, No. 1, pp. 182–196.

2. Tamir, A. An $O(pn^2)$ algorithm for the p -median and related problems on tree graphs. *Operations Research Letters*, 1996, Vol. 19, No. 1, pp. 59–64.
3. Brimberg, J., Juel, H., Schöbel, A. Linear Facility Location in Three Dimensions – Models and Solution Methods. *Operations Research*, 2002, Vol. 50, No. 6, pp. 1050–1057.
4. Gusev, A. I. Why do all the Roads Lead to Moscow? *World of Transport and Transportation*, Vol. 3, 2005, Iss. 4, pp. 22–25.
5. Gusev, S. A. Theorems of Resource Allocation. *World of Transport and Transportation*, Vol. 8, 2010, Iss. 3, pp. 30–35.
6. Gusev, A. I., Gusev, S. A. Optimal Location of Rescue Service. *World of Transport and Transportation*, Vol. 15, 2017, Iss. 4, pp. 194–201.
7. Gusev, A. I., Gusev, S. A., Milevsky, A. S. Provision of Optimal Transit Service under Even Distribution of Facilities. *World of Transport and Transportation*, Vol. 15, 2017, Iss. 6, pp. 32–46.
8. Gusev, S. A. Sitting of Rescue Services near One-Sided Transport Hub. *World of Transport and Transportation*, Vol. 16, 2018, Iss. 3, pp. 208–218. ●

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