ON THE PROBLEM OF NONLINEAR WAVES IN RODS

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ABSTRACT

The article shows new, nonclassical dispersion waves and vibrations, and their spectrum is studied for two reasons: firstly, there is an increase in the number of composite materials used in railway transport, which leads to the phenomenon of dispersion, and secondly, the spring of 2017 showed that many kilometers of tracks were flooded with shallow water. The task of detecting a track defect and determining its spatial localization can be solved with the help of measuring the energy of longitudinal, vertical and transverse oscillations. But the phenomenon of dispersion makes it very difficult to detect the occurrence of new track defects from the rolling stock (see World of Transport and Transportation, 2016, Vol. 14, Iss. 3). The method of «unfixed constructive change of variables» is used in the paper. The authors conclude that two necessary conditions for the solvability of a system of differential equations with first-order partial derivatives have one common nontrivial factor; and that through designing new exact solutions of the non-linear KdV equation that differ from the classical ones, it is found that there is a change of variables in which the equation for the function Y(ξ, d) = Y0(U(ξ, d), ξ) is separate from all the others.

Keywords: railway track, vibration spectrum, diagnostic system, Korteweg–de Vries equations, method of non-fixed constructive change of variables.

Background. *One of the points of view regarding creation in the future and functioning of a high-tech complex of quality control of the state of the railway track directly from a moving train and the theoretical foundations of the control method are set forth in the article [1], and the opinion of other authors on this topic is given in the works quoted in [1]. This refers to the idea of using that part of the energy of motion that goes into the energy of vibrations to search for defects. Let's recall that it is proposed to divide all railway tracks into sections of various lengths, taking into account the infrastructure and facilities, geology, etc. [1]. On each «serviceable» (usable) section, the reference spectrum of oscillations is recorded in the database, which is later compared with the current similar data. That is, the task of detecting a track defect and determining its spatial localization can be solved with the help of the energy of longitudinal, vertical and transverse oscillations that arise when a rolling stock passes, and it refers to problems that are called «inverse» in mathematics. Since there is a huge variety of boundary and initial conditions due to various geometric, climatic, and geological factors, then «inverse» problems even with only two spatial variables still require large resources, extraordinary computing powers, and at present there is no way to solve them.*

Solutions of «direct» problems with a single spatial variable are simpler and much less expensive. Mathematical methods and modern technologies allow one to obtain information about oscillations and waves, their amplitudes, spectrum envelopes, resonance frequencies, etc., which is an important clue when analyzing the spectrum of oscillations in each section of the railway track. Currently, the task of data accumulation, decoding of spectra and interpretation of possible causes of defects, identification of specific features, creation of databases of big data is of great importance (it is possible to draw an analogy with Big Data and Blockchain technologies).

The paper is focused on identifying new steps towards tracks' high-tech control. In some ways, if we speak in practical terms, this approach approximates creation of a prototype of a closed network of Russian Railways, which will implement the Digital Railway program, the principles of the Internet of Things [2]. Automatic systems will be built, computer networks of physical objects (things) equipped with built-in communication technologies that can make network requests without human intervention, monitor the functioning of other systems, automatics and robots, send data on the state of the railway tracks from remote regions of the Russian Federation to regional databases of Russian Railways from remote areas using the GLONASS system. The consolidation of regional computer centers of Russian Railways into a common network will allow for the search and mathematical analysis, without human participation, of data, which are similar in spectral and other characteristics, to identify causes and assess risks. It will further allow to implement practical solutions regarding each case.

Objective. *The objective of the authors is to consider new approaches to solution of the problem of nonlinear waves in rods, focusing on dispersion waves and oscillations, with regard to practical implementation of results for development of new track control systems for railways.*

Methods. *The authors use general scientific and mathematical methods, comprising mathematical models developed by Rayleigh, Love, Bishop, Mindlin, Herrmann, Korteweg-de Vries equations.*

Results. *The mathematical models proposed by Raileigh [John William Strutt, 3rd Baron Rayleigh] and Love [Augustus Edward Hough Love], Bishop [Richard Evelyn Donohue Bishop], Mindlin [Raymond David Mindlin], and Herrmann [George Herrmann] for rods and plates are considered in the scientific treatise [4], and it is shown that in certain conditions (the presence of composite materials, the Young's modulus) for describing longitudinal waves in rods, a nonlinear differential third-order partial differential equation of Korteweg–de Vries (KdV) is applicable.*

Setting a task

Remark 1. *Let's note that the idea that it is possible to use the desired function as an independent variable under certain conditions, goes back to the thesis of the great Russian mathematician S. A. Chaplygin [7].*

Interest in tasks related to the KdV equation

$$\frac{\partial Z(x,t)}{\partial t} + Z^n \frac{\partial Z(x,t)}{\partial x} + \beta \frac{\partial^3 Z(x,t)}{\partial x^3} = 0, \qquad (1)$$

was also manifested long ago within the school of academician V. P. Maslov [3].

The history of the problem and some methods for equation (1) are given in [3, 5, 6].

For n = 1 (weak dispersion) we have the classical nonlinear KdV equation, which is connected by the Miura transformation with the modified KdV equation *for n = 2 (the case of strong dispersion) [3, 4].*

Remark 2*. It is known from these works that the group of shear transformations $Z(x, t) = y(\theta)$ with the invariant $\theta = x - Vt$ is accompanied by an ordinary differential equation (ODE). Its first integral describes the oscillations of an inharmonic oscillator with nonlinearity* $y''(\theta) = -\dfrac{y(\theta)^{n+1}}{\beta(n+1)} + \dfrac{V\,y(\theta)}{\beta} - \dfrac{C_1}{\beta}, \quad C_1 = const.$

The order of the equation decreases, and this means that the oscillation equation ***separates****, that is, the first derivative depends only on the function $y(\theta)$.*

This conclusion is trivial in the case of one variable.

We multiply the last equation by $y'(\theta)$. Integrating the last equation, we obtain:

$$y'(\theta)^2 = E - \frac{2y(\theta)^{n+2}}{\beta(n+1)(n+2)} + \frac{V\,y(\theta)^2}{\beta} - 2\frac{C_1}{\beta}y(\theta),$$

where the integration constant E has the meaning of the initial energy, and the remaining terms on the right-hand side have the meaning of the potential energy with the negative sign. Hence follows the solution of K. Jacobi. From the point of view of the method of unfixed constructive change of variables (UFCCV), this is a degenerate case, since the Jacobian is zero.

However, there is another variant of manifestation of symmetry properties, in which the order of the equation does not decrease, and the equation for the first derivative and the desired function ***separates*** *from all others. This case is described below.*

Remark 3*. The application of UFCCV method allows one to find variables in which the nonlinear equation is written as a system of linear algebraic functional equations (SLAFE), and this makes it possible, firstly, to obtain new exact or approximate solutions, and, secondly, to construct dynamic systems, opening a possibility to study stability [3].*

New way of analysis

We assume that all functions are thrice continuously differentiable with respect to all variables. In (1) we make an arbitrary change of variables:

$$Z(x,t)\,\big|_{x=x(\xi,\delta),\ t=t(\xi,\delta)} = U(\xi,\delta)\,. \tag{2}$$

The inverse replacement restores the solution Z(x, t) of the equation with respect to the function $U(\xi, \delta)$, at least locally:

$$Z(x,t) = U(\xi,\delta)\,\big|_{\xi=\xi(x,t),\delta=\delta(x,t)}. \tag{3}$$

Let's suppose that the Jacobiant that is the determinant of the Jacobi matrix $J = \begin{pmatrix} x'_\xi & x'_\delta \\ t'_\xi & t'_\delta \end{pmatrix}$ *is not equal to zero or to infinity, i. e.* $\det J = x'_\xi t'_\delta - x'_\delta t'_\xi \neq 0, \infty.$

Then there is (again at least locally) the inverse transformation $\xi = \xi(x, t)$, $\delta = \delta(x, t)$. The inverse Jacobi matrix is $J^{-1} = \begin{pmatrix} \xi'_x & \xi'_t \\ \delta'_x & \delta'_t \end{pmatrix}.$

The ratio $JJ^{-1} = E$ must be satisfied, then we obtain formulas for recalculating the derivatives of the old variables $x(\xi, \delta)$, $t(\xi, \delta)$ with respect to the new variables $\xi(x, t)$, $\delta(x, t)$:

$$\frac{\partial x}{\partial \xi} = \det J\,\frac{\partial \delta}{\partial t}, \quad \frac{\partial t}{\partial \xi} = -\det J\,\frac{\partial \delta}{\partial x},$$
$$\frac{\partial x}{\partial \delta} = -\det J\,\frac{\partial \xi}{\partial t}, \quad \frac{\partial t}{\partial \delta} = \det J\,\frac{\partial \xi}{\partial x}. \tag{4}$$

Let's introduce the notation (we shall establish differential connections):

$$\frac{\partial Z}{\partial x}\,\Big|_{x=x(\xi,\delta),t=t(\xi,\delta)} = Y(\xi,\delta),$$
$$\frac{\partial Z}{\partial t}\,\Big|_{x=x(\xi,\delta),t=t(\xi,\delta)} = T(\xi,\delta). \tag{5}$$
$$\frac{\partial Y(\xi(x,t),\ \delta(x,t))}{\partial x}\,\Big|_{x=x(\xi,\delta),t=t(\xi,\delta)} = M(\xi,\delta).$$

Using (4) and (5), we obtain:

$$-\frac{\partial U}{\partial \xi}\frac{\partial x}{\partial \delta} + \frac{\partial U}{\partial \delta}\frac{\partial x}{\partial \xi} = T(\xi,\delta)(x'_\xi t'_\delta - t'_\xi x'_\delta), \tag{6}$$
$$\frac{\partial U}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial U}{\partial \delta}\frac{\partial t}{\partial \xi} = Y(\xi,\delta)(x'_\xi t'_\delta - t'_\xi x'_\delta), \tag{7}$$
$$\frac{\partial Y}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial Y}{\partial \delta}\frac{\partial t}{\partial \xi} = M(\xi,\delta)(x'_\xi t'_\delta - t'_\xi x'_\delta). \tag{8}$$

The equation (1) takes the form

$$\frac{\partial M}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial M}{\partial \delta}\frac{\partial t}{\partial \xi} =$$
$$= -(T(\xi,\delta) + U^n\,Y(\xi,\delta))\,(x'_\xi t'_\delta - t'_\xi x'_\delta)/\beta. \tag{9}$$

It is necessary to satisfy the ratio

$$\frac{\partial^2 Z(x(\xi,\delta),t(\xi,\delta))}{\partial x\,\partial t}\,\Big|_{x=x(\xi,\delta),\ t=t(\xi,\delta)} =$$
$$= \frac{\partial^2 Z(x(\xi,\delta),\ t(\xi,\delta))}{\partial t\,\partial x}\,\Big|_{x=x(\xi,\delta),\ t=t(\xi,\delta)} \tag{10}$$

in variables ξ, δ. This ratio, taking into account equations (3), (4), can be written in the form

$$-\frac{\partial Y}{\partial \xi}\frac{\partial x}{\partial \delta} + \frac{\partial Y}{\partial \delta}\frac{\partial x}{\partial \xi} - \frac{\partial t}{\partial \delta}\frac{\partial T}{\partial \xi} + \frac{\partial t}{\partial \xi}\frac{\partial T}{\partial \delta} = 0\,. \tag{11}$$

The system (6), (7), (9), (11) can be considered as a system of functional linear algebraic equations (SFLAE) with respect to variables (derivatives of old variables with respect to new ones) $\dfrac{\partial x}{\partial \delta}, \dfrac{\partial x}{\partial \xi}, \dfrac{\partial t}{\partial \xi}, \dfrac{\partial t}{\partial \delta}$ *, and it has a unique nontrivial explicit solution. This fact underlies the UFCCV method.*

We denote the partial derivatives of the function:

$$\frac{\partial x}{\partial \xi} = z_1(\xi,\delta), \quad \frac{\partial x}{\partial \delta} = z_2(\xi,\delta),$$
$$\frac{\partial t}{\partial \xi} = z_3(\xi,\delta), \quad \frac{\partial t}{\partial \delta} = z_4(\xi,\delta). \tag{12}$$

Let's introduce the notation for brevity of the notation for the nonlinear system (6), (7), (9), (11):

$$a_{12} = -\frac{\partial U}{\partial \xi} = -a_{24}, \quad a_{11} = \frac{\partial U}{\partial \delta} = -a_{23},$$
$$a_{32} = -\frac{\partial Y}{\partial \xi} = -a_{44}, \quad a_{43} = -\frac{\partial Y}{\partial \delta} = -a_{31},$$
$$a_{53} = -\frac{\partial M}{\partial \delta},\ a_{54} = \frac{\partial M}{\partial \xi},\ a_{33} = \frac{\partial T}{\partial \delta},$$
$$a_{34} = -\frac{\partial T}{\partial \xi},\ \mathrm{T}_1 = \mathrm{T} + \mathrm{U}^n\,\mathrm{Y}. \tag{13}$$

Theorem 1. *Suppose that we have SFAE – system of functional algebraic equations (6), (7), (9), (11). This system is* ***linear*** *(SFLAE), and there exists a unique nontrivial solution of the system of four equations:*

$$\frac{\partial x}{\partial \xi} = (a_{32}(a_{12}a_{53} - a_{11}a_{54})\beta T + a_{12}((a_{11}a_{34} - a_{12}a_{33})T_1 + $$
$$+ (a_{54}a_{33} - a_{53}a_{34})\beta Y)) / \Psi_0 = z_1(\xi, \delta), \qquad (14)$$
$$\Psi_0 = T((a_{11}a_{32} - a_{12}a_{31})T_1 + (a_{31}a_{54} - a_{32}a_{53})\beta Y) +$$
$$+ Y((a_{12}a_{33} - a_{11}a_{34})T_1 + (a_{53}a_{34} - a_{54}a_{33})\beta Y);$$

$$\frac{\partial x}{\partial \delta} = (a_{31}(a_{11}a_{54} - a_{12}a_{53})\beta T + a_{11}((a_{12}a_{34} - a_{11}a_{33})T_1 + \qquad (15)$$
$$+ (a_{53}a_{33} - a_{54}a_{34})\beta Y)) / \Psi_0 = z_2(\xi, \delta);$$

$$\frac{\partial t}{\partial \xi} = ((a_{11}a_{32} - a_{12}a_{31})(a_{12}T_1 - a_{54}\beta Y)) / \Psi_0 = z_3(\xi, \delta); \quad (16)$$

$$\frac{\partial t}{\partial \delta} = ((a_{11}a_{32} - a_{12}a_{31})(a_{11}T_1 - a_{53}\beta Y)) / \Psi_0 = z_4(\xi, \delta). \quad (17)$$

The Jacobian is then:
$detJ = \beta(a_{12}a_{31} - a_{11}a_{32}) \cdot (a_{12}a_{53} - a_{11}a_{54})) / \Psi_0 =$
$= g(\xi, \delta)$. (18)

The function $M(\xi, \delta)$ is calculated from (8):
$M(\xi, \delta) = (-a_{44}z_4(\xi, \delta) + a_{43}z_3(\xi, \delta)) / g(\xi, \delta)$. (19)

***Proof.** The system (6), (7), (9), (11) looks **complicated, nonlinear**, but in fact, as it is proved in this theorem, it is SFLAE. In [5–6], for a similar SFLAE. obtained for quasilinear parabolic equations, it is proved that there exists a unique nontrivial solution that is also constructed there explicitly. We follow the basic algorithm indicated in it. The solution of the system (6), (7), (9), (11) is first constructed in accordance with the well-known Gauss method.*

Two linear equations follow from three equations (6), (7), (9). For example, let's multiply (6) by $Y(\xi, \delta)$, and (7) by $T(\xi, \delta)$ and subtract the resulting equations. We obtain linear functional algebraic equation (LFAE) with respect to the variables (12). We carry out analogous elementary transformations, say, for equations (7), (9), and we obtain the second LFAE for the unknown functions (12). The equation (11) is linear with respect to the variables (12).

Thus, by elementary transformations we normalize three rows of the matrix to an almost triangular form. We express any three derivatives of the old variables by the new ones (12) through the fourth. We substitute the obtained ratios into the remaining equation, and here we are in for a surprise: the final equation turns out to be **linear!** *Then we obtain (14)–(17). Next, we calculate (18), (19). The theorem 1 is proved.*

If we consider (14)–(17) as a system of differential equations with partial derivatives, then the equality of the second mixed derivatives must necessarily be fulfilled [3].

***Theorem 2.** Let the equation KdV (1) and formulas for the change of variables (2)–(5) be given. Then the equation has the form SFLAE:*

$$a_{11}\frac{\partial x}{\partial \xi} + a_{12}\frac{\partial x}{\partial \delta} = T(\xi, \delta)\, g(\xi, \delta); \qquad (20)$$

$$a_{23}\frac{\partial t}{\partial \xi} + a_{24}\frac{\partial t}{\partial \delta} = Y(\xi, \delta)\, g(\xi, \delta); \qquad (21)$$

$$a_{31}\frac{\partial x}{\partial \xi} + a_{32}\frac{\partial x}{\partial \delta} + a_{33}\frac{\partial t}{\partial \xi} + a_{34}\frac{\partial t}{\partial \delta} = 0; \qquad (22)$$

$$a_{53}\frac{\partial t}{\partial \xi} + a_{54}\frac{\partial t}{\partial \delta} = -T_1(\xi, \delta)\, g(\xi, \delta) \qquad (23)$$

and of the equation (19).

Here the Jacobian (18) $g(\xi, \delta)$ is explicitly computed, and the function Ψ_0 is defined in (14).

The solution of the linear system SFLAE (20)–(23) has the form (14)–(17).

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***Proof.** We substitute the computed value of the Jacobian (18) into the right-hand side of (6), (7), (9), (11) and obtain the SFLAE (20)–(23).*

Thus, we have the first nonlinear dynamical system consisting of (14), (16), and also the second nonlinear dynamical system consisting of (15), (17), and two corresponding Jacobi matrices. The variables ξ, δ are treated as a new time, a parameter along the trajectories of the dynamic system. Using the theory of dynamical systems, one can carry out a study of stability of the solution [3]. An analysis of the properties of the main Jacobian is also carried out in [3].

Let's consider a particular case, *which simplifies the calculations considerably, namely, we set $t(\xi, \delta) = \xi$. We choose the second bracket in (17) and equate it to zero. We get*

$$\frac{\partial M}{\partial \delta} = -\frac{\partial U}{\partial \delta}T_1 \,/\, (\beta Y). \qquad (24)$$

After substituting in (19), (7) and simplifications, we have

$$M(\xi, \delta) = \frac{\partial Y}{\partial \delta}Y \,/\, \left(\frac{\partial U}{\partial \delta}\right),$$
$$x'_\delta(\xi, \delta) = U'_\delta(\xi, \delta) / Y(\xi, \delta). \qquad (25)$$

We differentiate (25) with respect to the variable δ and, equating to (24) considering (13), we obtain:

$$T(\xi, \delta) = -U''Y - \beta\left(\frac{\partial Y}{\partial \delta}\right)^2 Y \,/\, \left(\frac{\partial U}{\partial \delta}\right)^2 - \frac{\beta Y^2}{U'_\delta}\frac{\partial}{\partial \delta}\left(\frac{Y'_\delta}{U'_\delta}\right) = \qquad (26)$$
$$= -U''Y - \frac{\beta Y}{U'_\delta}\frac{\partial}{\partial \delta}\left(\frac{YY'_\delta}{U'_\delta}\right).$$

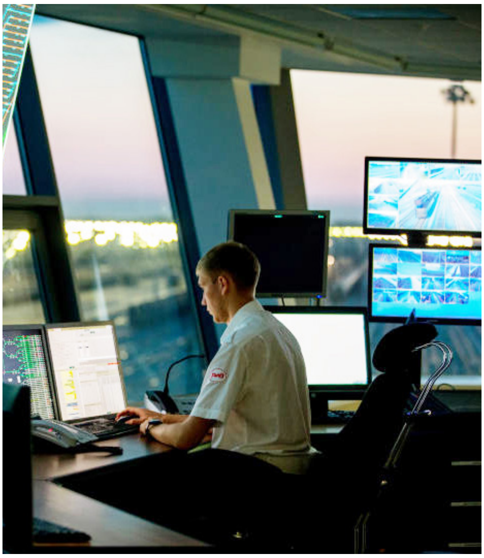
We calculate the first derivatives of the functions (25), (26) and the elements of the Jacobian matrix, then find the determinant, the eigenvalues and the trace of this matrix (divergence), which are criteria in theorems establishing stability in [3]. Let's note that according to what we know, no one has previously considered the KdV equation from such a point of view. Formally, there are two conditions for the solvability of a system (14)–(17). Two equalities of mixed derivatives have the form

$$\frac{\partial z_1(\xi, \delta)}{\partial \delta} - \frac{\partial z_2(\xi, \delta)}{\partial \xi} = T(\xi, \delta)\, Q(\xi, \delta) = 0,$$
$$\frac{\partial z_3(\xi, \delta)}{\partial \delta} - \frac{\partial z_4(\xi, \delta)}{\partial \xi} = Y(\xi, \delta)\, \Psi_o^2\, Q(\xi, \delta) = 0. \qquad (27)$$

There exists a nontrivial common factor $Q(\xi, \delta)$ for any function $M(\xi, \delta)$. Two solvability conditions (27) reduce to one relation
$Q(\xi, \delta) = 0$. (28)

New accurate solutions of the KdV equation

***Theorem 3.** Let SFLAE (20)–(23) and its solution (14)–(17) be given, and also the expressions for the*



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Jacobian (18) and the function $M(\xi, \delta)$ (19) be determined.

a) Then the following relations hold:

$$t(\xi,\delta)=\xi, \quad \frac{\partial M}{\partial \delta}=-\frac{\partial U}{\partial \delta} T_1(\xi,\delta) \;/\,(\beta Y),$$

$$\mathrm{T}_1(\xi,\delta)=\mathrm{T}(\xi,\delta)+\mathrm{U}^n(\xi,\delta)\;\mathrm{Y}(\xi,\delta),$$

$$M(\xi,\delta)=\frac{\partial Y}{\partial \delta} Y \;/\,(\frac{\partial U}{\partial \delta}),$$

$$x'_\delta(\xi,\delta)=U'_\delta(\xi,\delta)\,/\,Y(\xi,\delta) \qquad (29)$$

and expression for the function T has the form (26).

The solution of SFLAE (14)–(17) after simplifications takes the form:

$$\frac{\partial x}{\partial \xi}=(Y\,T'_\delta\,U'_\xi-T\,U'_\delta\,Y'_\xi)\,/\,(Y\,(Y\,T'_\delta-T\,Y'_\delta))=z_1(\xi,\delta),$$

$$\frac{\partial x}{\partial \delta}=U'_\delta\,/\,Y=z_2(\xi,\delta); \qquad (30)$$

$$\frac{\partial t}{\partial \xi}=(Y'_\delta\,U'_\xi-Y'_\xi\,U'_\delta)\,/\,(T\,Y'_\delta-T'_\delta\,Y)=z_3(\xi,\delta)=1,$$

$$\frac{\partial t}{\partial \delta}=z_4(\xi,\delta)=0. \qquad (31)$$

The general factor under the conditions necessary for solvability (28) takes the form of a linear differential equation with partial derivative of first order:

$$T'_\delta-T(\xi,\delta)Y'_\delta\,/\,Y+\left(\frac{\partial Y}{\partial \delta}\frac{\partial U}{\partial \xi}\right)\,/\,Y-\left(\frac{Y'_\xi\,U'_\delta}{Y}\right)=0. \qquad (32)$$

b) Then equation (32) ensures the fulfillment of the first equation (31) and, as a consequence, the differential equation with partial derivative of the third order separates, which connects only the functions $Y(\xi, \delta)$, $U(\xi, \delta)$:

$$\frac{\partial}{\partial \delta}\left(\int \frac{Y'_\delta\,U'_\xi}{Y^2}\,d\delta-\int \frac{Y'_\xi\,U'_\delta}{Y^2}\,d\,\delta+C_0(\xi)\right)=$$

$$=\frac{\partial}{\partial \delta}\left(U^n(\xi,\delta)+\frac{\beta}{U'_\delta}\frac{\partial}{\partial \delta}\left(\frac{Y\,Y'_\delta}{U'_\delta}\right)\right). \qquad (33)$$

Proof. The role of the initial time is played by the variable $t(\xi, \delta) = \xi$. In addition, it is important that the first equation (31) $(Y'_\delta\,U'_\xi-Y'_\xi\,U'_\delta)\,/\,(T\,Y'_\delta-T'_\delta\,Y)=1$ is identically satisfied by equality (32).

The construction of the accurate solution of equation (1) by the UFCCV method reduces in this case to the solution of the equation (33), which is reflected in Theorem 4.

Theorem 4. Let (1)–(6) be valid. Then, the differential equation with partial derivative (33) for the function $Y(\xi, \delta)$ **separates** from all other equations and depends only on the function $Y(\xi, \delta)$ and its derivatives:

$$\frac{\partial}{\partial \delta}\left(\;U^n(\xi,\delta)+\frac{\beta}{U'_\delta}\frac{\partial}{\partial \delta}\left(\frac{Y\,Y'_\delta}{U'_\delta}\right)\;\right)=$$

$$=\left(Y'_\delta\,U'_\xi-Y'_\xi\,U'_\delta\right)/\,Y^2. \qquad (34)$$

Replacement

$$t\,(\xi,\delta)=\xi, \quad Y(\xi,\delta)=Y_0(U(\xi,\delta),\delta)=$$

$$=Y_0(\eta,\delta),\,\eta=U(\xi,\delta) \qquad (35)$$

leads to nonlinear differential equations with partial derivatives of the third order:

$$\frac{1}{\beta}\frac{\partial Y_0(\eta,\xi)}{\partial \xi}+\frac{\partial}{\partial \eta}\left(Y_0^3\,\frac{\partial^2 Y_0}{\partial \eta^2}\right)+\frac{2\,\eta^{n-1}\,n}{\beta}Y_0^2(\eta,\,\xi)=0. \qquad (36)$$

The substitution $Y_0(\eta,\,\xi)=\pm\sqrt{G(\eta,\xi)}$ in equation (36) leads to a nonlinear equation with paritial derivative of the third order:

$$\frac{sign}{\beta\,G(\eta,\xi)^{(3/2)}}\,\frac{\partial G(\eta,\,\xi)}{\partial\,\tau}+\frac{\partial^3 G(\eta,\xi)}{\partial \eta^3}+\frac{2\,\eta^{-1+n}\,n}{\beta}=0. \qquad (37)$$

Some of the formulas below are given for the value sign = 1 (for sign = –1, they are written in the same way). The Jacobian has two identical expressions $\det \mathrm{J}=-\,\mathrm{U}'_\delta(\xi,\delta)\,/\,Y_0(\eta,\xi)$ and $Y_0(\eta,\,\xi)=\sqrt{G(\eta,\xi)}$.

$$M\,(\xi,\delta)=G'_\eta(\eta,\xi)\,/\,2, \quad T(\xi,\delta)=$$

$$=-\frac{1}{2}\,\sqrt{G(\eta,\xi)}\,(2\,\eta^n+\beta\,G''_{\eta\eta}(\eta,\xi)), \qquad (38)$$

$$x\,(\xi,\delta)=X\left(\eta,\xi\right), \quad \frac{\partial X(\eta,\xi)}{\partial \xi}=\eta^n+\frac{1}{2}\beta\,G''_{\eta\eta},$$

$$\frac{\partial\,X(\eta,\xi)}{\partial\;\eta}=\frac{1}{\sqrt{G(\eta,\xi)}}, \quad \det J=\left(\eta\;G'_\xi\,\left(U'_\delta\,(\xi,\delta)\right)\right)/\,z_1,$$

$$z_1=G^2\,(\eta,\xi)\left(2\,n\,\eta^n+\beta\,\eta\,\frac{\partial^3\,G(\eta,\xi)}{\partial \eta^{\,3}}\right),$$

$$\frac{\partial^2\,X(\eta,\xi)}{\partial \eta\,\partial \xi}=\frac{\partial^2 X(\eta,\xi)}{\partial\,\xi\,\partial \eta}.$$

Proof is carried out by direct substitution.

Remark 4. Constructing a solution of the KdV equations by the UFCCV method means to determine the functions $Y(\xi, \delta)$, $T(\xi, \delta)$, $M(\xi, \delta)$, $x(\xi, \delta)$. If we define the function $Y(\xi, \delta)$ from (36) or (37), then all other functions from the list will be found. Then it is possible to return in some sense, not always explicitly, to the «parametric form», to the original variables (2), (5). The nonlinear KdV equation has a new property in the general case, and not only in the «degenerate case»: there exists a change of variables in which the equation for the function $Y(\xi, \delta) = Y(U(\xi, \delta), \xi)$ **separates** from all the others.

Example. Let's briefly describe a nontrivial new exact solution (37), and hence in accordance with Remark 4, and (1). For n = 1 the equation (37) has a solution $G(\eta, \xi) = \Xi(\theta)$ with the invariant $\theta = \eta + V\xi$.

Firstly, from (37) there follows an ordinary differential equation (ODE3), which we don't write here. The first integral of ODE3 has the form ODE4

$$\Xi^{''}(\theta) + sign\, 2V / (\beta \sqrt{\Xi(\theta)}) = -C_4 - 2\theta / \beta,$$

where C_4 is the constant, and the parameter sign takes one of two values, namely «plus» or «minus» one. This is a nonlinear ODE4 with the right-hand side.

Secondly, it is possible to multiply ODE4 by the derivative of the function $\Xi'(\theta)$.

After integration, we obtain a quadrature

$$(\Xi^{'}(\theta))^2 / 2 + C_5 + C_4\, \Xi(\theta) + 2\,(\theta\, \Xi(\theta) -$$

$$-\int \Xi(\theta)\, d\theta) / \beta - sign\, 4V \sqrt{\Xi(\theta)} / \beta = 0,$$

where C_5 is the constant.

Let $X(\eta, \xi) = X_1(\eta + V\xi)$. Then it follows that $X_1(\theta) =$ sign $\int (1/\sqrt{\Xi(\theta)})\, d\theta$. This is the implicit equation for computing the function $\Xi(\theta)$.

We note, in addition, that a nonautonomous dynamical system $\Xi'(\theta) = p(\theta)$, $p'(\theta) = -C_4 - 2\theta/\beta +$ sign $2V/(\beta \sqrt{\Xi(\theta)})$ follows from ODE4.

We calculate the derivative $d\theta/d\Xi(\theta) = 1/(\Xi'(\theta)) = 1/p(\theta)$, then the Jacobi matrix 2 takes the form

$$\begin{pmatrix} 0 & 1 \\ -2/(\beta\, p(\theta)) + sign\, V / (\beta\, \Xi^{3/2}(\theta)) & 0 \end{pmatrix}.$$

The stationary point is determined by the expressions $p = 0$, $\Xi(\theta) = 4V^2/(C_4\beta + 2\theta)^2$.

It follows that for different values of the parameters on the equator of the Poincaré sphere there can exist singular points: a center or a saddle. Further, the existence of bifurcations is analyzed. A detailed research shows that a new exact solution describes a wave asymmetric with a larger derivative at the leading edge, different from the Jacobi solution, briefly described in Remark 2 and in [3].

There is another solution (36). The equation in the range of the tensile group becomes ODE 6:

$$Y(\xi, \delta) = \eta^{-\frac{(2+n)}{3n}}\, \Psi(\theta), \quad \theta = \tau^{\frac{2}{3n}}\, U(\xi, \delta),$$

$$\frac{\partial}{\partial \theta}\left(\Psi^3(\theta)\Psi^{''}(\theta)\right) + \frac{n\,\theta^{n-1}\,\Psi(\theta)^2}{\beta} -$$

$$-\frac{(2+n)}{3n\beta}\Psi(\theta) + \frac{2}{3n\beta}\theta\,\Psi^{'}(\theta) = 0.$$

In this case, the Jacobian is not equal to zero. Obviously, a full study of this solution should be presented in a separate paper. We note that equations (36), (37) are new in the theory of the KdV equation.

***Remark 5.** For the Korteweg–de Vries–Burgers equation [3, 4], which differs from equation (1) by the second derivatives (they describe dissipation) added to it, there is no solution constructed by the UFCCV method, similar to that constructed in the theorem 4.*

In the proof of Theorem 4, we obtain the same equation (37) three times. This does not occur in the equation with the Korteweg–de Vries–Burgers dissipation, since different equations arise in the process of the proof, differing in terms connected with dissipation.

The possibility of constructing asymptotic solutions is described in [3].

Conclusions.

1. Dynamical systems for the KdV equations are constructed on the basis of the method of unfixed constructive change of variables. It is possible to formulate conditions for stability of phase trajectories and a decrease in the volume of the phase flow.

2. It is proved that two necessary conditions for the solvability of a system of differential equations with partial derivatives of the first order have one common non-trivial factor.

3. New exact solutions of the non-linear KdV equation that differ from the classical ones are constructed. It is found that there is a change of variables in which the equation for the function $Y(\xi, \delta) = Y_0(U(\xi, \delta), \xi)$ separates from all the others.

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