OPTIMAL LOCATION OF RESCUE SERVICE

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ABSTRACT

In papers (1–4), the problems of optimal location of a highway, optimal distribution of means between programs for traffic safety in transport are investigated. The article suggests a model for location of the car fleet (automobile fleet) in its dynamics and investigated the problem of optimal location of the rescue service on the highway in the best possible way if there are several transport nodes thereon. Model calculations can be used both for practical purposes and for further theoretical studies.

Keywords: safety, transport, normal distribution law, Laplace function, transport node, optimal location, rescue service.

Background. *At the time of the accident vehicles rescue services are urgently needed. With a long extension of railways, highways, they cannot be located quite often. The problem arises of the optimal distribution of such services on the highway. When servicing a certain road section, rescuers move to the site of the accident at different distances and, in addition, to one area more often, to others less frequently, so it becomes necessary to locate the rescue service so that the average arrival time is minimal. In this article, an economic-mathematical model of the optimal location of the rescue service is developed for servicing several transport nodes on the highway.*

Objective. *The objective of the authors is to consider optimal location of rescue service.*

Methods. *The authors use general scientific and engineering methods, comparative analysis, mathematical calculations, graph construction.*

Results.

Normal location of transport

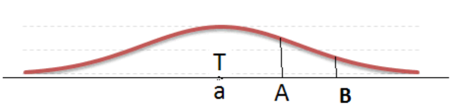
Let us have a transport node T (a) (station, marshalling base, etc.) on some specific main line, with some trains running along a dedicated main line, while others leave it in some places. Let us denote by I the number of cars that have passed through the node in a month (if a certain car passes several times, it is counted so many times, and these cars are assigned to this transport node, they are not assigned to another node in calculation).

Note that the selected time period does not affect the result. We will assume that the number of cars on the main line for the month is distributed according to the normal law (5) with the mathematical expectation a, the average quadratic deviation of σ and the intensity I (see Pic. 1), i.e. the number of cars that were on the section AB of the main line over a month is equal to

$$N=\frac{I}{\sqrt{2\pi}}\int_{A}^{B}e^{-\frac{(x-a)^2}{2\sigma^2}}dx.$$

The number of cars requiring maintenance of the rescue service on the AB section is proportional to the number of cars N. We are to place the rescue service so that its total costs for arriving for maintenance are minimal.

Let us save the rescue service C to the point c (Pic. 2).



Pic. 1. Normal location of transport near transport nodes.

Then the total costs of the service on arrival to the accident site will be proportional to:

$$A=\frac{I}{\sqrt{2\pi}\sigma}\int_{-\infty}^{c}(c-x)e^{-\frac{(x-a)^2}{2\sigma^2}}dx+\frac{I}{\sqrt{2\pi}\sigma}\int_{c}^{\infty}(x-c)e^{-\frac{(x-a)^2}{2\sigma^2}}dx.$$

So, we need to find c, for which A is minimal. In this case, with one transport node, the answer is obvious: c = a, but we will carry out a study on the optimality of the task in order to take advantage of the studies obtained when considering the problems in the presence of several transport nodes.

Optimum in the presence of one node

We make the change of variable x – a = y, (since the result does not depend on I, we set I = 1), we get:

$$A=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{c-a}(c-a-y)e^{-\frac{y^2}{2\sigma^2}}dy+$$

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{c-a}^{\infty}\left(y-(c-a)\right)e^{-\frac{y^2}{2\sigma^2}}dy=$$

$$=\frac{2}{\sqrt{2\pi}\sigma}\int_{c-a}^{\infty}\left(y-(c-a)\right)e^{-\frac{y^2}{2\sigma^2}}dy+$$

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(c-a-y)e^{-\frac{y^2}{2\sigma^2}}dy=$$

$$=c-a-2(c-a)\left(F\left(\frac{\infty}{\sigma}\right)-F\left(\frac{c-a}{\sigma}\right)\right)-$$

$$-\frac{2\sigma^2}{\sqrt{2\pi}\sigma}\int_{c-a}^{\infty}e^{-\frac{y^2}{2\sigma^2}}d\left(-\frac{y^2}{2\sigma^2}\right)=$$

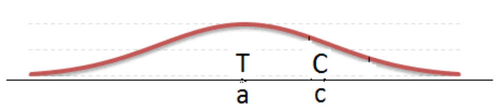
$$=c-a-2(c-a)\left(\frac{1}{2}-F\left(\frac{c-a}{\sigma}\right)\right)-\frac{2\sigma}{\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}}\Big|_{c-a}^{\infty}=$$

$$=2(c-a)F\left(\frac{c-a}{\sigma}\right)+\frac{2\sigma}{\sqrt{2\pi}}e^{-\frac{(c-a)^2}{2\sigma^2}},$$

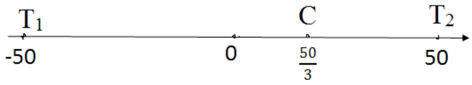
where $F(x)=\frac{1}{\sqrt{2\pi}}\int_{0}^{x}e^{-\frac{y^2}{2\sigma^2}}dy$ *– Laplace function.*

$$\frac{dA}{dc}=2F\left(\frac{c-a}{\sigma}\right)+\frac{2(c-a)}{\sqrt{2\pi}\sigma}e^{-\frac{(c-a)^2}{2\sigma^2}}-$$

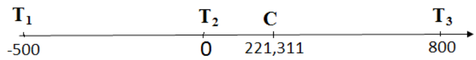
$$\frac{2(c-a)}{\sqrt{2\pi}\sigma}e^{-\frac{(c-a)^2}{2\sigma^2}}=2F\left(\frac{c-a}{\sigma}\right).$$



Pic. 2. Location of the rescue service on the main line.



Pic. 3. Optimal location of the rescue service for two transport nodes.



Pic. 4. Optimal location of the rescue service for three transport nodes.

Table 1

Finding the coordinates of the rescue service in increments of 10 (km)

c	F_1	F_2	$3F_1 + F_2$
-50	0	-0,47725	-0,47725
-40	0,039828	-0,46407	-0,34459
-30	0,07926	-0,4452	-0,20742
-20	0,117911	-0,41924	-0,06551
-10	0,155422	-0,38493	0,081335
0	0,191462	-0,34134	0,233043
10	0,225747	-0,28814	0,389096
20	0,258036	-0,22575	0,548362
30	0,288145	-0,15542	0,709012
40	0,31594	-0,07926	0,86856
50	0,341345	0	1,024034

Table 2

Finding the coordinates of the rescue service with an accuracy of 1 (km)

c	F_1	F_2	$3F_1 + F_2$
-20	0,117911	-0,41924	–0,06551
-19	0,12172	-0,41621	-0,05105
-18	0,125516	-0,41309	-0,03654
-17	0,1293	-0,40988	-0,02198
-16	0,133072	-0,40658	-0,00737
-15	0,136831	-0,4032	0,007292
-14	0,140576	-0,39973	0,022002
-13	0,144309	-0,39617	0,036761
-12	0,148027	-0,39251	0,05157
-11	0,151732	-0,38877	0,066428
-10	0,155422	-0,38493	0,081335

For c = a we have $\frac{dA}{dc}\ 0$: at c > a the derivative > 0, at c < a the derivative < 0. Hence it follows that the point a – minimum point, which was to be proved.

Optimum at two nodes

Suppose that there are two transport nodes $T_1(a_1)$ and $T_2(a_2)$ in the service section with the corresponding normal distribution laws for cars with parameters (a_1, σ_1) and (a_2, σ_2). The nodes' intensities are equal to I_1, I_2. It is necessary to find for them the optimal location of the rescue service C(c). Making similar calculations, as in the previous section, we get:

$$A = 2I_1(c-a_1)F\left(\frac{c-a_1}{\sigma_1}\right)+\frac{2I_1\sigma_1}{\sqrt{2\pi}}e^{-\frac{(c-a_1)^2}{2\sigma_1^2}}+$$

$$+2I_2(c-a_2)F+\frac{2I_2\sigma_2}{\sqrt{2\pi}}e^{-\frac{(c-a_2)^2}{2\sigma_2^2}}.$$

$$\frac{dA}{dc}=2I_1F\left(\frac{c-a_1}{\sigma_1}\right)+2I_2F\left(\frac{c-a_2}{\sigma_2}\right).$$

$$\frac{d^2A}{dc^2}=\frac{2I_1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(c-a_1)^2}{2\sigma_1^2}}+\frac{2I_2}{\sqrt{2\pi}\sigma_2}e^{-\frac{(c-a_2)^2}{2\sigma_2^2}}>0.$$

Therefore the function $\frac{dA}{dc}$ increases. Suppose for definiteness that $a_1 < a_2$, then at $c = a_1$ we have $\frac{dA}{dc}=2I_2F\left(\frac{a_1-a_2}{\sigma_2}\right)<0$, and at $c = a_2$ we get $\frac{dA}{dc}=2I_1F\left(\frac{a_2-a_1}{\sigma_1}\right)>0$, and therefore the equation $\frac{dA}{dc}=0$ has a unique solution on the interval (a_1, a_2), and at this point a minimum is reached. In a general case, an example of a solution of the equation $\frac{dA}{dc}=0$ with the indicated accuracy for Excel will be considered later, but now we will analyze a special case when the intensities of transport nodes are equal, i.e. $I_1 = I_2$. Then the equation $\frac{dA}{dc}=0$ is equivalent to the equation

$$F\left(\frac{c-a_1}{\sigma_1}\right)=F\left(\frac{a_2-c}{\sigma_2}\right).$$

Hence $\frac{c-a_1}{\sigma_1}=\frac{a_2-c}{\sigma_2}$ and get a formula for the optimal location of the rescue service:

$$c \quad \frac{a_1\sigma_2 \quad a_2\sigma_1}{\sigma_1 \quad \sigma_2}.$$

In particular, if $\sigma_2 = \sigma_1$, then $c=\frac{a_1+a_2}{2}$.

<u>Example 1.</u> Let's consider the problem in specific numbers. Let the intensities of the transport nodes T_1(–50 km), T_2(50 km) are the same; σ_1 = 100 km; σ_2 = 50 km. Here is the coordinate of the optimal location of the rescue service $c=\frac{50(100-50)}{150}=\frac{50}{3}$ (km).

Pic. 3 shows the schematic arrangement of transport nodes and rescue service.

Now let's take an example where the intensities of transport nodes are not equal.

<u>Example 2</u>. The intensity of the transport node T_1 is three times larger than T_2, i.e. $I_1 = 3I_2$. It is required to find the optimal coordinates of the rescue service with an accuracy of 1 meter.

Decision. The optimal coordinates are found from the equation:

$$\frac{dA}{dc}=2I_1F\left(\frac{c-a_1}{\sigma_1}\right)+2I_2F\left(\frac{c-a_2}{\sigma_2}\right)=0.$$

or

$$3F\left(\frac{c+50}{100}\right)+F\left(\frac{c-50}{50}\right)=0.$$

This equation cannot be solved analytically, but it is possible to find the optimal coordinate with any accuracy. And in many ways (using tools Excel).

We will do this in several steps.

(-50; 50) is divided in increments of 10 (km), at the points of the partition we calculate $F_1 = F\left(\frac{c+50}{100}\right)$, $F_2 = F\left(\frac{c-50}{50}\right)$, *$3F_1+F_2$. The results of the calculations are given in Table 1.*

It can be seen from the table that the optimal coordinate lies between -20 (km) and -10 (km). Now we divide the interval (-20, -10) in increments of 1 (km), at the points of the partition we calculate $F_1 = F\left(\frac{c+50}{100}\right)$, $F_2 = F\left(\frac{c-50}{50}\right)$, *$3F_1 + F_2$. The results of the calculations are given in Table 2.*

The table shows that the optimal coordinate lies between -16 (km) and -15 (km). Continuing the same way three more times, we will find the optimal coordinate of the rescue service with an accuracy of 1 meter. We get c = -15497 meters.

Optimum for more than two nodes

Let there be several transport nodes $T_1(a_1)$, $T_2(a_2)$,..., T_k with the corresponding normal distribution laws of cars with parameters (a_1, σ_1), (a_2, σ_2),..., (a_k, σ_k). The intensities of nodes are equal to I_1, I_2,...I_k. It is necessary to find the optimal location of the rescue service C(c) for each of them. Using the previous calculations, we get that the total costs of the rescue service C will be proportional:

$$A = 2I_1(c-a_1)F\left(\frac{c-a_1}{\sigma_1}\right) + \frac{2I_1\sigma_1}{\sqrt{2\pi}}e^{-\frac{(c-a_1)^2}{2\sigma_1^2}} + $$
$$+2I_2(c-a_2)F\left(\frac{c-a_2}{\sigma_2}\right) +$$
$$+\frac{2I_2\sigma_2}{\sqrt{2\pi}}e^{-\frac{(c-a_2)^2}{2\sigma_2^2}} + \ldots + 2I_k(c-a_k)F\left(\frac{c-a_k}{\sigma_k}\right) +$$
$$+\frac{2I_k\sigma_k}{\sqrt{2\pi}}e^{-\frac{(c-a_k)^2}{2\sigma_k^2}}.$$

$$\frac{dA}{dc} = 2I_1F\left(\frac{c-a_1}{\sigma_1}\right) + 2I_2F\left(\frac{c-a_2}{\sigma_2}\right) + \ldots + 2I_kF\left(\frac{c-a_k}{\sigma_k}\right).$$

$$\frac{d^2A}{dc^2} = \frac{2I_1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(c-a_1)^2}{2\sigma_1^2}} + \frac{2I_2}{\sqrt{2\pi}\sigma_2}e^{-\frac{(c-a_2)^2}{2\sigma_2^2}} + \ldots$$
$$+\frac{2I_k}{\sqrt{2\pi}\sigma_k}e^{-\frac{(c-a_k)^2}{2\sigma_k^2}} > 0.$$

Hence it follows that the function $\frac{dA}{dc}$ *increases. Suppose for definiteness that $a_1 < a_2 <\ldots< a_k$, then at $c = a_1$ we have* $\frac{dA}{dc} < 0$, *and at $c = a_k$, we get* $\frac{dA}{dc} > 0$. *Therefore, for the equation* $\frac{dA}{dc} = 0$ *a unique solution on the interval (a_1, a_k) is possible, and at this point a minimum is reached. And it is here that it is rational to place a rescue service. Let's examine the supporting example.*

Example 3. Let there be three transport nodes T_1(-500 km), T_2(0), T_3(800 km), respectively σ_1 = 800 km, σ_2 = 400 km, σ_3 = 300 km. The intensities of all are the same. It is necessary to find the optimal coordinates of the rescue service with an accuracy of 1 meter.

The solution is related to the equation

$$F\left(\frac{c-a_1}{\sigma_1}\right) + F\left(\frac{c-a_2}{\sigma_2}\right) + F\left(\frac{c-a_3}{\sigma_3}\right) = 0.$$

Substituting the numerical data, we get:

$$F\left(\frac{c+500}{800}\right) + F\left(\frac{c}{400}\right) + F\left(\frac{c-800}{300}\right) = 0.$$

Further, we proceed as in example 2. As a result, we find that c = 183297 meters.

Example 4. Let's consider the previous example, in which the intensities of the transport nodes are 1:3:2.

The solution reduces to the equation

$$F\left(\frac{c-a_1}{\sigma_1}\right) + 3F\left(\frac{c-a_2}{\sigma_2}\right) + 2F\left(\frac{c-a_3}{\sigma_3}\right) = 0.$$

Substituting the numerical data, we get:

$$F\left(\frac{c+500}{800}\right) + 3F\left(\frac{c}{400}\right) + 2F\left(\frac{c-800}{300}\right) = 0.$$

We proceed similarly to example 2. Result: c = 221311 meters. Pic. 4 shows the schematic arrangement of transport nodes and rescue service.

Conclusion. *A technique has been developed for the optimal placement of the rescue service when servicing several transport nodes distributed according to the normal law. Equations are found for finding the coordinates of the rescue service. Examples and corresponding numerical computations are given with the help of Excel tools.*

The above methods of selecting options can be used in calculating the location of the rescue service on the railway main line, determining the optimal location of motor vehicle stops, other important objects, and also in further theoretical studies.

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